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Non-critical string duals of $\mathcal{N} = 1$ quiver theories

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Abstract: We construct $\mathcal{N}$ =1 non-critical strings in four dimensions dual to strongly coupled $\mathcal{N}$=1 quiver gauge theories in the Coulomb phase, generalizing the string duals of Argyres-Douglas superconformal fixed points in $\mathcal{N}$=2 gauge theories. They are the first examples of superstring vacua with an exact worldsheet description dual to chiral $\mathcal{N}$=1 theories. We identify the dual of the non-critical superstring using a brane setup describing the field theory in the classical limit. We analyze the spectrum of chiral operators in the strongly coupled regime and show how worldsheet instanton effects give non-perturbative information about the gauge theory. We also consider aspects of D-branes relevant for the holographic duality.

Keywords: Gauge-gravity correspondence, D-branes, Conformal Field Models in String Theory, Brane Dynamics in Gauge Theories.

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Contents

1. Introduction

Four dimensional little string theories (LST), or four-dimensional non-critical strings [1 – 3], are very important examples of holographic dualities in string theory. Unlike the usual $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence [5], the string background has no Ramond-Ramond fluxes and is often exactly solvable.[1] They are gravitational duals of non-local theories whose infrared dynamics is described by four-dimensional $\mathcal{N}=2$ gauge theories. In opposition with type

[1] However, some of the most interesting examples, dual to confining $\mathcal{N}=1$ gauge theories [4], don't have a known exact worldsheet description.

IIB six-dimensional LST [6 − 8, 1], flowing in the infrared to free $\mathcal{N} = (1, 1)$ SYM$_6$, they probe the gauge theory far from the semi-classical regime, near a strongly coupled fixed point. These models can be viewed either as coming from NS5-branes wrapped on a Seiberg-Witten curve [9] or equivalently from the decoupling limit of CY$_3$ singularities [1]. In both cases the infrared limit of the dual, non-gravitational theory is an $\mathcal{N} = 2$ gauge theory in a Coulomb phase, and more specifically for the examples that we study pure $\mathcal{N} = 2$ SYM theory in the neighborhood of a strongly coupled superconformal fixed point, of the Argyres-Douglas type [10]. More recently, these non-critical strings were related to $\mathcal{N} = 1$ SYM at large large N [11].

From the string theory point of view, these models can be studied using exact worldsheet conformal field theory methods. It is fortunate that we have such a worldsheet CFT description since the string theory, being non-critical, does not have a good supergravity limit, i.e. with gradients of the background fields small w.r.t. the string scale. However, in the regime where the holographic duality is valid we focus on the neighborhood of a superconformal fixed point of the gauge theory, for which the coupling constant of the string dual diverges. Then, when the singularity is resolved in order for the string coupling to be finite, one cannot have a large hierarchy between the mass scales of the effective gauge theory and the little string theory scale while keeping the string coupling small, and the gravitational dual will encode little string theory physics which is not field-theoretical. Nevertheless the far infrared physics of the theory, corresponding to localized excitations in the bulk, will still be well described by gauge theory. We argue, in our particular example, that the worldsheet instanton effects — related to the duality between the supersymmetric coset $SL(2, \mathbb{R})/U(1)$ and the $\mathcal{N} = 2$ Liouville theory (called also FZZ duality) [12, 8] — indeed contain the information about the exact Seiberg-Witten low energy effective action the gauge theory, i.e. the masses of the BPS dyons as functions of the coordinates on the moduli space of $\mathcal{N} = 2$ SYM theory.

We will show in this work how to extend this construction to $\mathcal{N} = 1$ models, that are non-critical string duals of $\mathcal{N} = 1$ chiral gauge theories in the Coulomb phase, near a strongly coupled superconformal fixed point. The gauge/string dual pair that we consider here provides an example of $\mathcal{N} = 1$ holography that can be trusted even for a small number of colors — in opposition with AdS$_5$/CFT$_4$ when one needs to take a 't Hooft large N limit because only the supergravity regime is available — and the worldsheet instantons effects are fully included in the abstract algebraic description of the worldsheet theory. Indeed the string theory vacua, being obtained as an asymmetric orbifold of the $\mathcal{N} = 2$ model, is still free of Ramond-Ramond fluxes, and the worldsheet theory still exactly solvable.

The type of gauge theories that we will obtain are quiver gauge theories, i.e. arrays of gauge groups with bifundamental matter. Theories of this sort have been introduced some time ago [13], an appeared more recently in the context of D-branes on orbifold singularities [14]. The particular models considered in this paper were first studied in [15], and then obtained in string theory using an NS5-brane/D-brane/orbifold setup [16]. They can be also found using a T-dual "brane box" model [17], as well as some D-branes on singularities constructions [18]. These field theories have been found also to be useful to "deconstruct" higher dimensional gauge theories [19].

JHEP04(2006)029

We will study the gauge theory in the semi-classical regime — using a boundary worldsheet CFT involving D-branes suspended between NS5-branes at orbifold singularities — and then identify the $\mathcal{N}=1$ four-dimensional non-critical string related by holography to an Argyres-Douglas superconformal field theory that can be found in the moduli space of the Coulomb phase of the quiver. We give the explicit mapping between the gauge-theory chiral operators and massless string states near the superconformal fixed points of the quiver theories. The monopoles and dyons that would become massless at the superconformal fixed point correspond in the string theory to non-supersymmetric D-branes, which are nevertheless stable.

We start in section 2 by reviewing the construction of $\mathcal{N}=2$ non-critical superstrings in four dimensions, giving their partition function and the massless spectrum. In section 3 we discuss the duality with $\mathcal{N}=2$ gauge theories near a superconformal fixed point and study the localized D-branes corresponding to the light dyons. Then in section 4 we explain how to construct four-dimensional vacua of type II non-critical strings with $\mathcal{N}=1$ supersymmetry. The dual gauge theory is discussed in section 5, as well as the localized D-branes. Finally in the conclusion we discuss some generalizations that will be the object of a forthcoming paper. In the appendix are gathered some facts about characters of $\mathcal{N}=2$ worldsheet superconformal field theories.

2. Four dimensional non-critical superstrings with $\mathcal{N}=2$ supersymmetry

Four dimensional non-critical superstrings are defined [2] as $\mathcal{N}=2$ linear dilaton backgrounds with four-dimensional Poincaré invariance. A large class of those can be built out of

$$\mathbb{R}^{3,1} \times \mathbb{R}_Q \times U(1) \times \left.\frac{SU(2)}{U(1)}\right|_{k_1} \times \left.\frac{SU(2)}{U(1)}\right|_{k_2}, \tag{2.1}$$

where k_1 and k_2 are arbitrary integers bigger or equal to two, corresponding to the levels of the super-cosets $SU(2)/U(1)$, the minimal models of the $\mathcal{N}=2$ superconformal algebra. The background charge Q of the linear dilaton is fixed by the cancellation of the conformal anomaly to be

$$Q = \sqrt{\frac{1}{k_1} + \frac{1}{k_2}} \tag{2.2}$$

and the $U(1)$ factor is requested for space-time supersymmetry, since it will play the role of the $U(1)$ R-symmetry in spacetime.

In this note we will consider the special case $k_2 = 2$ and $k_1 = n$, i.e. with only one minimal model (the super-coset SU(2)/U(1) at level two contains just the identity and thus trivializes)

$$\mathbb{R}^{3,1} \times \mathbb{R}_Q \times U(1) \times \left.\frac{SU(2)}{U(1)}\right|_{n}. \tag{2.3}$$

The background charge of this linear dilaton background is then given by

$$Q = \sqrt{\frac{n+2}{n}}. \tag{2.4}$$

Note that the superstring theories (2.3) are intrinsically non-critical and their effective action receives large α' corrections. In the special case $n = 2$ there are no $SU(2)/U(1)$ factors and the string theory is the well-known worldsheet CFT mirror to the conifold [20, 21]. We will study the more generic situation (2.1) in a following publication.

To construct a spacetime-supersymmetric background out of these $\mathcal{N} = 2$ SCFT building blocks we need to perform a generalized GSO projection, i.e. to keep only states (in the light-cone gauge) with odd-integral left and right worldsheet R-charge [22]. Before this orbifoldization, the left and right R-charges are given as follows

$$
\begin{aligned}
Q_R &= Q_{\text{FER}} + \sqrt{1+\frac{2}{n}}\, p^{\text{X}}_{\text{L}} - \frac{m}{n} \quad \text{mod } 2\,, \\
\bar{Q}_R &= \bar{Q}_{\text{FER}} + \sqrt{1+\frac{2}{n}}\, p^{\text{X}}_{\text{R}} + \frac{\bar{m}}{n} \quad \text{mod } 2\,,
\end{aligned}
\tag{2.5}
$$

in terms of the left and right momenta $p^{\text{X}}_{\text{L,R}}$ along the $U(1)$ free field X, and $m/2$, $\bar{m}/2$ the left and right $U(1) \subset SU(2)$ eigenvalues in the $\mathcal{N} = 2$ coset $SU(2)/U(1)|_n$. Note that $\bar{m}$ enters in the R-charge of the right-movers with opposite sign compared to m for the left-movers. The charges Q_{FER} and $\bar{Q}_{\text{FER}}$ are the ordinary contributions form the fermion number in the NS or in the R sector. Clearly the compact boson X has to be compactified to a specific radius and twisted to obtain a GSO-projected theory.

2.1 The one-loop partition function

The information about the spectrum can be read out of the one-loop vacuum amplitude. A consistent spectrum has to be compatible with the modular invariance of this partition function on the torus. We will concentrate on the contribution to the partition function of delta function-normalizable operators, i.e. those which propagate along the non-compact linear dilaton direction ρ,

$$
\mathcal{O}_P(z,\bar{z}) = e^{-Q(\frac{1}{2}+iP)\rho(z,\bar{z})}\ \mathcal{Y}(z,\bar{z})\,,
\tag{2.6}
$$

where $\mathcal{Y}(z,\bar{z})$ is the vertex operator corresponding to the other degrees of freedom, including the superconformal ghosts. The non-normalizable operators are also important and we will discuss them below, however they don't enter (by construction) in the one-loop vacuum amplitude.

The spectrum of the delta-normalizable operators is insensitive to the physics in the strong coupling region $\rho \to -\infty$, in particular to the presence of a potential that would regularize it (as we will consider later), even though the sub-leading term in the density of states depend upon this potential, as can be shown using a path-integral approach to the partition function [23–26]. We are interested in the following in the type IIA superstring theory, for which the four-dimensional non-critical string will be dual to $\mathcal{N} = 2$ gauge theory in a way that we will discuss below. The modular-invariant partition function for

JHEP04(2006)029

the four-dimensional non-critical strings studied in this paper is given by (see also [25])

$$
\begin{aligned}
&Z_{\text{CONT}}(\tau,\bar\tau) \qquad (2.7)\\
&= \frac{1}{4\pi^2\alpha'\tau_2}\frac{1}{\eta^2(\tau)\bar\eta^2(\bar\tau)}\ \frac{1}{4}\sum_{a,b,\bar a,\bar b\in\mathbb{Z}_2}\ \sum_{\{\upsilon_\ell\},\{\bar\upsilon_\ell\}\in(\mathbb{Z}_2)^4}(-)^{a+\bar a+b(1+\sum_i\upsilon_i)+\bar b(1+\sum_i\bar\upsilon_i)+\bar a\bar b}\ \times\\
&\times\frac{\Theta_{a+2\upsilon_1,2}(\tau)\ \bar\Theta_{\bar a+2\bar\upsilon_1,2}(\bar\tau)}{\eta(\tau)\bar\eta(\bar\tau)}\sum_{2j=0}^{n-2}\sum_{m,\bar m\in\mathbb{Z}_{2n}}C^{j\,(a+2\upsilon_3)}_{m}(\tau)\ \bar C^{j\,(\bar a+2\bar\upsilon_3)}_{\bar m}(\bar\tau)\ \times\\
&\times\int_0^\infty \mathrm{d}P\sum_{r\in\mathbb{Z}_{n+2}}Ch_c^{(a+2\upsilon_2)}(P,2m+n(a+2\upsilon_4)+4nr;\tau)\ \times\\
&\times\bar{Ch}_c^{(\bar a+2\bar\upsilon_2)}(P,-2\bar m+n(\bar a+2\bar\upsilon_4)+4n\bar r;\bar\tau)\ \delta_{m-a-2\upsilon_4-4r,\bar m+\bar a+2\bar\upsilon_4+4\bar r \mod 2(n+2)}\,.
\end{aligned}
$$

One can check that this one-loop amplitude is modular-invariant and has the required properties (spin-statistics, worldsheet local $\mathcal{N}=1$ superconformal symmetry and locality w.r.t. the spacetime supercharges). The partition function is expressed in terms of $C^{j\,(s)}_m$, the characters of the $\mathcal{N}=2$ minimal model $SU(2)/U(1)|_n$. The contribution of the linear dilaton direction and the compact U(1) free field $J=i\partial X$, as well as their fermionic superpartners, have been recast together in terms of $Ch^{(s)}_c(P,m)$, the extended characters of $\mathcal{N}=2$ superconformal algebra with $c=3+3Q^2$. The main properties of these characters are gathered in the appendix. The left and right worldsheet R-charges of this model, defined by eq. (2.5), are integral for all the states propagating in this one-loop amplitude, and the final projection giving a set of mutually local vertex operators is ensured by the $\mathbb{Z}_2$ projection acting on the fermion number, similar to superstrings in flat ten-dimensional spacetime. Thus this is a spacetime supersymmetric model, with one supersymmetry from the left-movers and one from the right-movers. To write explicitly the spacetime supercharges we can bosonize the worldsheet fermions of the $\mathbb{R}_Q\times U(1)$ theory as

$$
\xi^\pm(z)=\frac{\xi^\rho(z)\pm i\xi^{\mathrm{X}}(z)}{\sqrt2}=e^{\pm iH_2(z)}\quad,\qquad \tilde\xi^\pm(\bar z)=\frac{\tilde\xi^\rho(\bar z)\pm i\tilde\xi^{\mathrm{X}}(\bar z)}{\sqrt2}=e^{\pm i\tilde H_2(\bar z)}\,. \tag{2.8}
$$

Then the spacetime supercharges of the gravitational theory come from the gravitino vertex operators. However when one considers the non-gravitational dual of this string theory it is natural to consider instead the four-dimensional supercharges as given by (in the $(-1/2,0)$ picture)

$$
\begin{aligned}
\mathcal{Q}^{\mathrm{L}}_\alpha(z) &= \oint \mathrm{d}z\ e^{-\frac{\phi(z)}{2}}\ \mathcal{S}^{\mathrm{L}}_\alpha(z)\ e^{+i\frac{Q}{2}X_L(z)}\ e^{+\frac{i}{2}H_2(z)}\ V^{(1,0)}_{0,\frac12,0}(z,\bar z)\,,\\
\bar{\mathcal{Q}}^{\mathrm{L}}_{\dot\alpha}(z) &= \oint \mathrm{d}z\ e^{-\frac{\phi(z)}{2}}\ \bar{\mathcal{S}}^{\mathrm{L}}_{\dot\alpha}(z)\ e^{-i\frac{Q}{2}X_L(z)}\ e^{-\frac{i}{2}H_2(z)}\ V^{(3,0)}_{0,-\frac12,0}(z,\bar z)\,,
\end{aligned}
$$

in terms of the $SL(2,\mathbb{C})$ spin fields $\mathcal{S}^{\mathrm{L}}_\alpha$ and $\bar{\mathcal{S}}^{\mathrm{L}}_{\dot\alpha}$ from the left-moving worldsheet SCFT. We denote by $V^{(s,\bar s)}_{j,m/2,\bar m/2}$ a primary of the coset $[SU(2)_{n-2}\times SO(2)_1]/U(1)$ with quantum numbers $(j,m,\bar m,s,\bar s)$, see the appendix for conventions. These supercharges are non-normalizable operators in the bulk. The spacetime supercharges from the right-moving sector are constructed in the same manner:

$$\mathcal{Q}^{\rm R}_{\alpha}(\bar z) = \oint {\rm d}\bar z \; e^{-\frac{\tilde\phi(\bar z)}{2}} \; \mathcal{S}^{\rm R}_{\alpha}(\bar z) \; e^{-i\frac{Q}{2}X_R(\bar z)} \; e^{-\frac{i}{2}\tilde H_2(\bar z)} \; V^{(0,3)}_{0,0,+\frac12}(z,\bar z)\,,$$
$$\bar{\mathcal{Q}}^{\rm R}_{\dot\alpha}(\bar z) = \oint {\rm d}\bar z \; e^{-\frac{\tilde\phi(\bar z)}{2}} \; \bar{\mathcal{S}}^{\rm R}_{\dot\alpha}(\bar z) \; e^{+i\frac{Q}{2}X_R(\bar z)} \; e^{+\frac{i}{2}\tilde H_2(\bar z)} \; V^{(0,1)}_{0,0,-\frac12}(z,\bar z)\,, \tag{2.9}$$

with now the spin fields from the right-movers. Overall it leads to $\mathcal{N}=2$ supersymmetry in four dimensions.

2.2 Massless spectrum

In order to obtain precise statements about the holographic duals of these models, we will study the spectrum of string operators which are massless in space-time. The allowed quantum numbers for the minimal models $SU(2)/U(1)|_n$ and for the $U(1)$ compact boson can be read from the partition function given by eq. (2.8).

We start with the massless states of this type IIA non-critical superstring in the NS-NS sector. First we find the universal hypermultiplet and the graviton multiplet from the delta-function normalizable operators of the linear dilaton. Then, we are looking for massless states with zero worldsheet R-charge from the four-dimensional spacetime part, that are non-normalizable along the linear dilaton direction. These operators have support in the weak coupling region, and are believed to be dual to off-shell, gauge-invariant operators in the four-dimensional gauge theory. We need to glue the massless states from the left- and right-moving sectors in a way compatible with the one-loop amplitude (2.8). We can construct worldsheet chiral primaries of dimension $h = Q/2 = 1/2$ by starting with a chiral primary of $SU(2)/U(1)|_n$ of dimension $h = {}^1\!/_2 - {}^{j+1}\!/_n$, coming from a state in an $SU(2)$ representation of spin j with $m = 2(j+1)$ and $s_3 = 2$, i.e. fermion number one. For the right moving sector we can choose an anti-chiral primary with $(\bar m = 2(j+1), \bar s_3 = 2)$. Such a state is compatible with the partition function and corresponds to the vertex operator (in the $(-1,-1)$ picture):

$$\mathcal{V}^{(c,a)}_{j} = e^{-\varphi-\tilde\varphi} \; e^{ip_\mu X^\mu} \; e^{-Q\tilde\jmath\rho} \; e^{i\frac{2(j+1)}{\sqrt{n(n+2)}}(X_L - X_R)} \; V^{(2,2)}_{j,\,j+1,\,j+1}\,. \tag{2.10}$$

One can first define this massless operator on-shell, from the four-dimensional point of view, in a way giving a worldsheet chiral (c,a) primary, if we choose for the linear dilaton momentum

$$\tilde\jmath = \frac{2(j+1)}{n+2}\,. \tag{2.11}$$

Then it will be dual to an operator of the dual theory if it obeys the Seiberg bound [27], i.e. if it is non-normalizable:

$$\tilde\jmath < \frac12 \implies 4(j+1) < n+2\,. \tag{2.12}$$

By taking values of $\tilde\jmath$ different from (2.11) this operator can be defined off-shell (in the four-dimensional sense). There is another dressing of these operators, giving a physical state but which is not an worldsheet chiral primary, obtained by replacing $\tilde\jmath \to 1 - \tilde\jmath$ and with a Seiberg bound now given by

$$\tilde{\jmath} = \frac{n-2j}{n+2} \implies 2j > \frac{n}{2} + 1\,. \tag{2.13}$$

These massless operators are chiral in spacetime. From the worldsheet expression of the space supercharges (2.9) we find that they are all bottom components of $\mathcal{N}=2$ vector multiplets in spacetime. We can of course construct similarly (a,c) primary states. However there are no massless states in the spectrum that are worldsheet (c,c) or (a,a) primaries, as was already observed in [28], so no hypermultiplets in the dual gauge theory that are mapped to closed string states. The R-charge in spacetime of the (c,a) and (a,c) operators that we found can be defined as

$$R = \frac{2i}{Q}\oint(\partial X - \bar{\partial}X) = \frac{8(j+1)}{n+2}\,. \tag{2.14}$$

The R-charge is thus identified with the winding number w around the $U(1)$, as $w = Q^2R/4$. In the double scaling limit, this will correspond to winding around the cigar, which is not conserved. It is consistent with the breaking of the $U(1)_R$ R-symmetry in the gauge theory dual which is not superconformal in the double scaling limit.[2] Also the Cartan eigenvalue of the $SU(2)_R$ symmetry is given by the momentum through, for a generic operator

$$m_{\mathrm{R}} = \frac{i}{Q}\oint(\partial X + \bar{\partial}X) = \frac{a+\bar{a}}{2} + \upsilon_4 + \bar{\upsilon}_4 \mod 2\,. \tag{2.15}$$

This charge spectrum is similar to the Cartan of a diagonal $SO(3)$ from the $SO(3)_1\times SO(3)_1$ algebra of 3 left-moving and 3 right-moving fermions; however the other $SO(3)$ generators are not affine symmetries of the worldsheet theory. The NS-NS chiral operators that we are considering are singlets of this $SU(2)_R$, as expected since they are bottom component of vector multiplets.

It is expected on general grounds that these string operators, that are massless, chiral in spacetime and not normalizable (i.e. they satisfy the Seiberg bound), are dual to chiral off-shell operators in the dual non-gravitational theory. Giving a (constant) vacuum expectation value (VEV) to these operators should be achieved by taking the *normalizable branch* of the same operators, i.e. the branch violating the Seiberg bound, at zero spacetime momentum. Then, since no spacetime supersymmetry should be broken by these VEVs, the corresponding worldsheet operator has to be $\mathcal{N}=2$ worldsheet chiral. These considerations show that the dual chiral operators in spacetime are given (on-shell) by the branch (2.13), and their VEVs are given by considering the operators in the branch (2.11) violating the Seiberg bound, i.e. for the $SU(2)$ spins $j > {}^{n-2}/_{4}$.

For completeness we study the operators the R-R sector. We have to look simply for the Ramond ground states of each of the $\mathcal{N}=2$ SCFT factors of the string theory. Let's take to begin an $SU(2)/U(1)_n$ coset. The Ramond ground states are given by $(m=2j+1, \upsilon_3=0)$

[2] This convention is consistent with the fact that, with the selection rule that appears in the partition function (2.8), only the (c,a) and the (a,c) states are part of the spectrum. One could instead construct a consistent model if we kept only (c,c) and (a,a); these two choices correspond to T-dual models. Our choice for the R-charge (2.14) as a winding number is consistent with keeping only (c,a) and (a,c) states.

JHEP04(2006)029

or $(m = -2j-1, \upsilon_3 = 1)$ which are respectively the one-half spectral flow of a chiral primary and of an anti-chiral one. For the $[\mathbb{R}_Q \times U(1)]$ factor one can have also a Ramond ground state for $\upsilon_2 = 0$ if $p_{\mathrm{X}} > 0$, or and $\upsilon_2 = 1$ if $p_{\mathrm{X}} < 0$, which would correspond respectively to the one-half spectral flow of a chiral primary and of an anti-chiral one, provided we choose $\tilde{\jmath}$ accordingly. However not all the combinations appear in the spectrum of string theory. As for the supercharges are given by (2.9), only the operators with $\upsilon_2 = \upsilon_3 = \upsilon_4$ (and similarly $\bar{\upsilon}_2 = \bar{\upsilon}_3 = \bar{\upsilon}_4$) appear. Then we obtain massless states in the string theory given by (in the $(-1/2, -1/2)$ picture)

$$\begin{aligned}\mathcal{U}_j = C_{\mu\nu} e^{-\frac{\phi+\tilde{\phi}}{2}} e^{-Q\tilde{\jmath}\rho} \Big\{ & \mathcal{S}^{\mathrm{L}}_{\alpha} \sigma^{\mu\nu\ \alpha}_{\ \ \ \beta}\, \mathcal{S}^{\mathrm{R}\ \beta} e^{iQ\left[\frac{2j}{n+2}+\frac{1}{2}\right](X_L(z)-X_R(\bar{z}))}\ e^{\frac{i}{2}(H_2(z)-\tilde{H}_2(\bar{z}))}\ V^{(1,3)}_{j,j+\frac{1}{2},j+\frac{1}{2}} \\ & + \ \bar{\mathcal{S}}^{\mathrm{L}}_{\dot{\alpha}} \bar{\sigma}^{\mu\nu\ \dot{\alpha}}_{\ \ \ \dot{\beta}}\, \bar{\mathcal{S}}^{\mathrm{R}\ \dot{\beta}} e^{-iQ\left[\frac{2j}{n+2}+\frac{1}{2}\right](X_L(z)-X_R(\bar{z}))}\ e^{-\frac{i}{2}(H_2(z)-\tilde{H}_2(\bar{z}))}\ V^{(3,1)}_{j,-j-\frac{1}{2},-j-\frac{1}{2}} \Big\}\,,\end{aligned} \tag{2.16}$$

where $C_{\mu\nu}$ is a polarization for the field strength of the vector field. These operators are indeed R-R ground states if we choose for the Liouville momentum $\tilde{\jmath} = \frac{2j}{n+2}$. The self- and anti-self-dual parts have $U(1)_R$ charge

$$R = \pm \left[2 + \frac{8j}{n+2} \right]\,, \tag{2.17}$$

and are singlets of $SU(2)_R$, as expected.

2.3 Resolution of the singularity

To obtain a manageable perturbative description of this non-critical string, one needs to regularize the strong coupling of the linear dilaton. From the worldsheet point of view it amounts to add an appropriate potential to the worldsheet action, such that the D-branes localized in the strong coupling region become all massive.

We expect that the worldsheet chiral operators that we discussed that obey the Seiberg bound will be dual to chiral operators in the spacetime holographic theory. They have support in the weak coupling region $\rho \to \infty$. We have seen that in our four-dimensional LST the NS-NS physical operators that satisfy the Seiberg bound are given by (2.10). Giving VEVs to these operators can be achieved, as we discussed above, if one add to the worldsheet action the conjugate of such an operator under $\tilde{\jmath} \to 1 - \tilde{\jmath}$, which has support in the strong coupling region $\rho \to -\infty$ and violates the Seiberg bound. For such an operator not to break spacetime supersymmetry it has to be a worldsheet chiral operator of the $\mathcal{N} = 2$ superconformal algebra. The first example from (2.10) is the "$\mathcal{N} = 2$ Liouville operator", given by

$$\mathcal{V}^{(c,a)}_{\frac{n}{2}-1} = e^{-\varphi-\tilde{\varphi}}\ e^{ip_\mu X^\mu}\ e^{-Q\tilde{\jmath}\rho} e^{i\sqrt{\frac{n}{n+2}}(X_L - X_R)} V^{(2,2)}_{\frac{n}{2}-1,\frac{n}{2},\frac{n}{2}}\,, \tag{2.18}$$

Giving a vacuum expectation value to the dual of this operator in the spacetime theory is then achieved by adding this operator in the first branch (2.11), for which it is a worldsheet (c, a) state of the $\mathcal{N} = 2$ algebra, in the $(0, 0)$ picture (and its conjugate):

$$\delta\mathcal{L} = \mu_{\mathrm{LIOUVILLE}} G^-_{-1/2} \tilde{G}^+_{-1/2}\ e^{-\sqrt{\frac{n}{n+2}}[\rho - i(X_L - X_R)]}\ +\ \text{c.c.}\,, \tag{2.19}$$

using the field identification in $\mathcal{N} = 2$ minimal models $(j, m, s) \sim (\frac{n}{2} - j - 1, m + n, s + 2)$.

After turning on this exactly marginal worldsheet deformation, the strong coupling region is regularized in a way that preserve $\mathcal{N} = 2$ superconformal symmetry. Indeed, if we build an $\mathcal{N} = 2$ worldsheet twisted chiral superfield whose bottom component is $\phi = \rho - i(X_L - X_R)$, we can write this potential as a twisted F-term

$$\delta\mathcal{L} = \mu_{\text{LIOUVILLE}} \int \mathrm{d}\bar{\theta}^- \mathrm{d}\theta^+ \; e^{-\frac{\Phi}{Q}} \; + \; \text{c.c.}\,. \tag{2.20}$$

There is another type of exactly marginal worldsheet perturbation, preserving $\mathcal{N} = 2$ superconformal symmetry on the worldsheet, that can be added to the Lagrangian. We can consider the following dressing of the identity in the $\mathcal{N} = 2$ minimal models:

$$\delta\mathcal{L} = \mu_{\text{CIGAR}} \; G^+_{-1/2} \tilde{G}^+_{-1/2} \; \xi^- \tilde{\xi}^- \; e^{-\sqrt{\frac{n+2}{n}}\rho}\,. \tag{2.21}$$

The $(0,0)$ picture vertex operator added to the Lagrangian is then a supersymmetric descendant of a worldsheet (a,a) operator. However in opposition with the previous type of operators, it is physical only for the specific value of the Liouville momentum $\tilde{\jmath} = 1$, which violates the Seiberg bound; thus this normalizable operator cannot be viewed as holographically dual of a vacuum expectation value for some gauge theory observable. It we consider the theory already perturbed by the $\mathcal{N} = 2$ potential, see eq. (2.20), we can nevertheless consider a normalizable operator with those quantum numbers. It is so possible to turn on a marginal perturbation with this asymptotic expansion. As for the $\mathcal{N} = 2$ Liouville potential, it can be also written in terms of the $\mathcal{N} = 2$ twisted chiral superfield as a D-term

$$\delta\mathcal{L} = -\mu_{\text{CIGAR}} \int \mathrm{d}^4\theta \; e^{-\frac{Q}{2}(\Phi + \Phi^*)} \tag{2.22}$$

Up to total derivatives, this operator is the first term in the asymptotic expansion of the supersymmetric sigma model with a cigar geometry

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \tanh^2(\rho/\sqrt{\alpha' k})\; \mathrm{d}X^2\,, \tag{2.23}$$

with $k = \frac{2n}{n+2}$. This cigar geometry is the target space metric of the gauged super-WZW model $SL(2,\mathbb{R})/U(1)$ at level k [29–31]. This coset model has also a non-trivial dilaton profile, which is bounded from above and asymptotically linear

$$\Phi = \Phi_0 + \frac{1}{2}\log k - \log\cosh(\rho/\sqrt{\alpha' k})\,, \tag{2.24}$$

such that the coefficient of the perturbation (2.21) of the linear dilaton background can be related to the zero mode of the dilaton in the cigar theory as $\mu_{\text{CIGAR}} = 4e^{-2\phi_0}/k$. Consequently the coefficient of the cigar perturbation is related to the string coupling at the tip of the cigar

$$g^2_{\text{EFF}} = k e^{2\Phi_0} = \frac{4}{\mu_{\text{CIGAR}}}\,. \tag{2.25}$$

However we should be aware that, because of the GSO projection, the cigar geometry is not meaningful in the four-dimensional non-critical string that we consider; in particular, the background for the target space contains non-zero flux for the NS-NS two-form since it represents a wrapped NS5-brane.

To summarize we can regularize the strong coupling regime of the linear dilaton by adding the $\mathcal{N}=2$ Liouville potential and/or the cigar perturbation. It is known [48] that a the consistent $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ SCFT requires that both perturbations are present at the same time, with related coefficients (we will come back later to this important issue). Then we consider our four-dimensional non-critical strings with the replacement $\mathbb{R}_Q \times S^1_{\mathrm{X}} \longrightarrow \mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)|_{\frac{2n}{n+2}}$, giving the double scaling limit of four-dimensional little string theory [8]. It is known that in the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ CFT the winding around the compact direction is not conserved; this intuitive fact is related algebraically to the fact that free-field computations of the correlators involve the insertion of the $\mathcal{N}=2$ Liouville interaction (2.20) as screening charges, thus violating winding conservation by integer amounts. However in our string construction, because of the GSO projection, the winding is fractional, leaving a conserved $\mathbb{Z}_n$ charge. This will be matched precisely with the $U(1)_R$ anomaly of the gauge theory. Note finally that they are various more complicated regularizations of the strong coupling regime (by turning on other deformations), however we don't have an explicit solution of the resulting worldsheet conformal field theory because they have a non-trivial $SU(2)/U(1)$ dependence.

2.4 Discrete spectrum in the double scaling limit

As we already mentioned, this $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$-based background contains not only delta-normalizable states, given by the partition function (2.8), and non-normalizable states discussed above, but also a discrete spectrum of normalizable states. They can be thought as bound states living near the tip of the cigar. The contribution of these discrete states is requested for modular-invariance of the partition function and reads (see [25, 26, 28] for more details):

$$
\begin{aligned}
Z_{\mathrm{DISC}} = & \frac{1}{4\pi^2\alpha'\tau_2}\frac{1}{\eta^2\bar{\eta}^2}\ \frac{1}{4} \sum_{\{\upsilon_\ell\},\{\bar{\upsilon}_\ell\}\in(\mathbb{Z}_2)^4}\ \sum_{a,b,\bar{a},\bar{b}} (-)^{a+\bar{a}+b(1+\sum_i \upsilon_i)+\bar{b}(1+\sum_i \bar{\upsilon}_i)+\bar{a}\bar{b}} \frac{\Theta_{a+2\upsilon_1,2}\Theta_{\bar{a}+2\bar{\upsilon}_1,2}}{\eta\bar{\eta}} \\
& \sum_{2j=0}^{n-2}\ \sum_{m,\bar{m}\in\mathbb{Z}_{2n}}\ \sum_{r\in\mathbb{Z}_{n+2}} \int_{\frac{1}{2}}^{\frac{n}{n+2}+\frac{1}{2}} \mathrm{d}\tilde{\jmath}\quad \delta\left(\tilde{\jmath} - \frac{m-\bar{m}+2n(r+\bar{r})-a-\bar{a}+n(\upsilon_4+\bar{\upsilon}_4)}{2(n+2)} + \frac{\mathbb{Z}}{2}\right) \\
& \qquad Ch_d^{(a+2\upsilon_2)}(\tilde{\jmath}, 2m+n(a+2\upsilon_4)+4nr)\bar{Ch}_c^{(\bar{a}+2\bar{\upsilon}_2)}(\tilde{\jmath},-2\bar{m}+n(a+2\bar{\upsilon}_4)+4n\bar{r}) \ \times \\
& \qquad\qquad \times\ \delta_{m-a-2\upsilon_4-4r,\bar{m}+\bar{a}+2\bar{\upsilon}_4+4\bar{r} \mod 2(n+2)}\ C_m^{j\,(a+2\upsilon_3)}\ \bar{C}_{\bar{m}}^{j\,(\bar{a}+2\bar{\upsilon}_3)}
\end{aligned}
\tag{2.26}
$$

written in terms of the discrete extended characters of the supersymmetric $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset $Ch_d^{(s)}(j,m)$ defined in the appendix. Thanks to the δ-function that implements the constraint of the gauging the spectrum of representations of $\mathrm{SL}(2,\mathbb{R})$ spin $\tilde{\jmath}$ is discrete. In the massless sector it contains the normalizable states that we expect from our previous analysis, that correspond to VEVs for the observables dual to the worldsheet operators (2.10).

3. Holographic duals of $\mathcal{N}=2$ non-critical superstrings

In this section we will discuss the holographic interpretation of these 3+1-dimensional

JHEP04(2006)029

non-critical superstrings. We will focus on the examples corresponding to the A_n modular invariant (i.e. diagonal w.r.t. j) for the $\mathcal{N}=2$ minimal model $SU(2)/U(1)$ that enters in the string theory description detailed in the previous section. One can of course also consider the D and E series in a similar fashion.

3.1 Argyres-Douglas superconformal field theories and wrapped fivebranes

The low-energy effective action of an $\mathcal{N}=2$ $SU(n)$ gauge theory at a generic point in the moduli space of the Coulomb branch contains $(n-1)$ Abelian vector multiplets of $\mathcal{N}=2$. The moduli space is parameterized by the symmetric polynomials s_ℓ in the eigenvalues of ϕ, the scalar component of the gauge multiplet, obtained by the generating function

$$P_n(x) := \langle \det(x-\phi) \rangle = \sum_{\ell=0}^{n} s_\ell \, x^\ell \tag{3.1}$$

with $s_n = 1$ and $s_{n-1} = 0$ due to the tracelessness condition. In terms of those the couplings of the exact low energy effective action can be obtained from the periods of the the Seiberg-Witten curve (SW) [32] of the gauge theory, given by [33, 34]

$$\Sigma : \quad y^2 = \frac{1}{4} P_n(x)^2 - \Lambda^{2n} \tag{3.2}$$

where Λ is the dynamically generated scale. At some points the moduli space develops singularities, corresponding to new states becoming massless, magnetic monopoles and/or dyons. They enter in the effective action as hypermultiplets charged under some of the U(1) factors. In particular one can get a superconformal field theory if we choose the vacua given by

$$P_n(x) = x^n + v_c \quad \text{with} \qquad v_c = (-)^n \langle \det \phi \rangle = \pm 2\Lambda^n \tag{3.3}$$

called Argyres-Douglas (AD) fixed points [10]. The dyons that become massless are mutually non-local and we obtain a strongly coupled $\mathcal{N}=2$ superconformal theory. It is the most singular fixed point in all the moduli space of the gauge theory, and $n(n-1)/2$ dyons become massless [43]. These dyons are charged under only $\lfloor \frac{n-1}{2} \rfloor$ Abelian gauge multiplets, thus the additional ones decouple.

The superstring dual of such $\mathcal{N}=2$ gauge theories is obtained using a configuration of D4-branes suspended between NS5-branes in type IIA superstrings [35], that we will discuss in more detail in section 4.3 in order to construct the $\mathcal{N}=1$ models. An $SU(n)$ gauge theory corresponds to a pair of NS5-branes, branes, located say at $x^6 = 0$ and $x^6 = L$, with n D4-branes suspended between them.[3] The positions of the D4-branes in the complex plane $x = x^4 + ix^5$ parameterize the Coulomb branch of the theory. Then the one-loop effects in the gauge theory correspond to the bending of the NS5-branes by the endpoints of the D4-branes (see [37] for a worldsheet derivation). To include the instantons effects, one can lift the brane setup to M-theory, for which the system is described by one

[3] This configuration leads naively to a $U(n)$ gauge theory but the diagonal $U(1)$ — corresponding to the center of mass motion of the D4-branes in the (4,5) plane — is frozen.

M5-brane wrapping the Riemann surface of genus $n-1$

$$\Sigma\ :\ t+t^{-1}+\Lambda^{-2n}\,P_n\left(\frac{2\pi}{\alpha'}\,x\right)=0\,, \tag{3.4}$$

where $t=\exp(-s/R_{10})$, with $s=x^6+ix^{10}$, and the radius $R_{10}=\sqrt{\alpha'}g_s$ of the eleventh dimension gives (in string units) the type IIA string coupling constant. This curve is exactly the Seiberg-Witten curve of the theory, after the change of variables $t=\Lambda^{-n}[y-P_n(\frac{2\pi}{\alpha'}x)/2]$. When we approach one of the AD points of the gauge theory the Riemann surface is degenerate; thus the eleven-dimensional supergravity picture is not valid.

Instead we can as in [9] consider a type IIA NS5-brane wrapping the same curve Σ; this can be achieved e.g. by compactifying the transverse space of the M5-brane along another direction x^7, and then performing the Kaluza-Klein reduction along this circle rather than along x^{10} (the radius of the transverse circle to the NS5-brane becomes irrelevant in the decoupling limit that we will consider). The string theory becomes strongly coupled at the Argyres-Douglas point and thus the degrees of freedom at the singularity can be decoupled from gravity in the limit where the asymptotic string coupling constant g_s is sent to zero. From the worldsheet point of view this singular wrapped fivebrane solution corresponds to a four-dimensional non-critical superstring of the sort (2.3), that we studied in detail in the previous section, with an infinite throat along the linear dilaton direction. The dyons that become massless at the AD point corresponds to various D-branes "localized" at infinite string coupling in the linear dilaton direction.[4] As for the decoupling limit of flat NS5-branes this linear dilaton background is holographically dual to the decoupled theory living on the singularity; the worldvolume theory on the fivebranes is a non-local little string theory [6] which flows in the infrared for the reasons explained above to an Argyres-Douglas superconformal field theory.

To resolve the singularity corresponding to the Argyres-Douglas fixed point one can turn on VEVs for the gauge-invariant operators parameterizing the Coulomb branch away from their value at the fixed point, in a way giving mass to all the dyons. Then we can define a *double scaling limit* [8] of the dual string theory, where the theory is decoupled from gravity by taking the limit $g_s\to 0$, while keeping the masses of the D-branes holographically dual to the "light" dyons fixed. This limit keeps only the universal behavior near those singularities, i.e. various gauge theory will belong to the same universality classes; for example, the most singular AD fixed point for $SU(n)$ discussed above can be found in the moduli space of all $SU(N\geqslant n)$. The most symmetric of those deformations corresponds to the polynomial $P_n(x)=x^n+v_c+\delta v$, dual on the string theory side to adding an $\mathcal{N}=2$ Liouville potential. This specific deformation of the superconformal AD fixed point breaks $U(1)_R$ to $\mathbb{Z}_n$, matching precisely, as previously noted, the winding non-conservation in the cigar $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$.

[4]In this strong coupling region one would need to come back to the eleven-dimensional picture, where we expect to find an and AdS_5 solution (much like the 5+1-dimensional type IIA LST that is well described in the deep infrared by $AdS_7\times S^4$ [7]). In the present case it is more difficult to make such a picture precise because the string background is strongly curved — and even non-critical. However for the more generic models defined in (2.1) there's a sensible ten-dimensional gravitational interpretation, as we will show in a forthcoming publication, and we expect to find an $AdS_5\times X^6$ solution of eleven-dimensional supergravity.

We have to stress the fact that there is no decoupling between little string theory modes and gauge theory physics in the regime where the non-critical string dual is weakly coupled. The gauge theory, near the Argyres-Douglas point, has a low a low-energy scale of order[5] $m_{\rm DYON}$ and a high-energy scale Λ. The former is associated to the size of the cycles of the "small" torus, while the latter is associated to the cycles of the "large" torus of the Seiberg-Witten curve near the degeneration limit. The coupling constant of the gauge theory $\tau = \exp 2i\pi/n$ is independent of the separation of scales $m_{\rm DYON}/\Lambda$. In the string theory dual, the low energy scale is of the order of the D-brane mass. It is natural to associate the high energy scale Λ with the (little) string scale $1/\sqrt{\alpha'}$. At any rate the gauge theory scale cannot be much smaller since otherwise the singularities of the SW curve associated to the cycles of the large torus will be visible in the non-critical string dual. The coupling constant of the non-critical string dual is given by the ratio $g_{\rm EFF} \sim 1/\sqrt{\alpha'}m_{\rm D}$ of those two scales. Then, is we want that the dyons masses are low compared to the high energy scale Λ, the non-critical string dual is perturbative only for very low energies $E \ll m_{\rm D} = 1/\sqrt{\alpha'}g_{\rm EFF}$.

3.2 Chiral spectrum

The chiral spectrum obtained from the string theory dual can be read from the analysis in section 2.2. This special case of four-dimensional LST has been previously considered in [2]. The bottom components of spacetime-chiral operators in the gauge theory coupling to the bulk operators discussed above are given by the following holographic dictionary

$$\mathcal{V}_j^{(c,a)} = e^{-\varphi-\tilde{\varphi}}\ e^{ip_\mu X^\mu}\ e^{-Q\tilde{j}\rho}\ e^{i\frac{2(j+1)}{\sqrt{n(n+2)}}(X_L-X_R)}\ V^{(2,2)}_{j\,j+1\,j+1} \qquad \longleftrightarrow \qquad s_{2j+2}(\phi)\,, \tag{3.5}$$

where the symmetric polynomials appearing of the right-hand side are implicitly defined by eq. (3.1). However this correspondence is not exact, since there is some mixing between the operators appearing on the right-hand side. These effects have been computed in [36] in the context of six-dimensional LST. However the core of the computation had to do with the $SU(2)/U(1)$ part so they can be used also in the present context. The result is that the polynomials $s_r(\phi)$ have to be replaced by the following combination of multi-trace operators

$$s_r(\phi) \rightarrow \sum_{\ell=1}^{n}\sum_{r_i=2}^{r} \frac{1}{\ell!}\left(\frac{1-r}{n}\right)^{\ell-1} \delta_{\sum r_i, r}\ \frac{1}{r_1}{\rm Tr}\,(\phi^{r_1})\cdots\frac{1}{r_\ell}{\rm Tr}\,(\phi^{r_\ell})\,. \tag{3.6}$$

The two sets of operators agree only in the limit $\frac{r-1}{n} \rightarrow 1$. The R-charge in space-time of these chiral operators is given by the winding around the $U(1)$ as discussed previously:

$$R = \frac{8(j+1)}{n+2}\,. \tag{3.7}$$

[5] For the deformation of the singularity that we consider, all the dyons have masses of the same order (at finite n).

At the superconformal fixed point, the scaling dimension of these chiral primary fields can be obtained from the spacetime extended superconformal algebra as

$$\Delta = 2I + \frac{R}{2} \tag{3.8a}$$
$$= \frac{4(j+1)}{n+2}\,, \tag{3.8b}$$

where, in (3.8a), I is the spin under the $SU(2)_R$ symmetry which is zero for these operators. The allowed values of the spin j in the $SU(2)/U(1)$ coset are such that the string vertex operator obeys the Seiberg bound

$$2j+1 > \frac{n}{2}\,, \tag{3.9}$$

keeping half of the $n-1$ values of $2j+1 = 1,\ldots,n-1$ in the $\mathcal{N}=2$ minimal model, and then all the spacetime chiral primaries satisfy the unitarity bound $\Delta \geqslant 1$. The operator of highest dimension, i.e. for $2j+1 = n-1$, corresponds to the (non-normalizable branch of the) $\mathcal{N}=2$ Liouville potential. This string theory spectrum matches exactly the spectrum of relevant deformations of the superconformal fixed point predicted in the gauge theory [10], as it has been already observed in [2]. We can extend straightforwardly the dictionary to the Ramond-Ramond sector, giving the correspondence

$$\mathcal{U}_j \;\longleftrightarrow\; C_{\mu\nu}\mathrm{Tr}\left(F^{\mu\nu}\phi^{2j-1}\right), \tag{3.10}$$

again up to multitrace corrections. The correlators of these off-shell observables lead to poles corresponding to one-particle states created from the vacuum, see [36]. The poles from the R-R sector give $\lfloor\frac{n-1}{2}\rfloor$ Abelian gauge fields; the truncation on the number of $U(1)$ multiplets comes from the Seiberg bound. From the NS-NS sector we get the same number of neutral scalars. Together with the R-NS and NS-R sectors they form $\lfloor\frac{n-1}{2}\rfloor$ gauge multiplets of $\mathcal{N}=2$ as expected from the gauge theory; indeed the dyons that are massless at the critical point are not charged under the other Abelian gauge multiplets, which decouple from the strongly interacting superconformal field theory.

3.3 D-branes, dyons, and $\mathbf{SL(2,\mathbb{R})/U(1)} - \mathcal{N}=2$ Liouville duality

The light BPS dyons of the gauge theory — that would become massless at the Argyres-Douglas singularity — correspond in the doubly scaled little string theory to the localized D-branes of the cigar; similarly as we increase the string coupling constant at the tip (corresponding to approaching the superconformal fixed point) they become lighter. We will now consider the boundary worldsheet CFT description of these D-branes, similar to the analysis carried out in six-dimensional LST [37].

Our goal is to find the couplings between the closed string modes and the D-branes, in other worlds to study the associated boundary states. The most non-trivial part of the computation comes from the boundary state associated to the localized B-brane in the super-coset SL(2, $\mathbb{R}$)/U(1), which has been found only recently [38 − 40]. Then we need to combine it consistently with the contribution of an A-brane of the $\mathcal{N}=2$ minimal model $SU(2)/U(1)$ and of a D0-brane of the flat spacetime in the GSO-projected theory (see [38]

JHEP04(2006)029

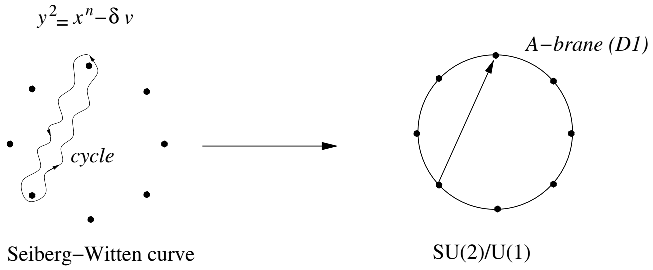

Figure 1: Periods of the SW curve and corresponding D-branes in $SU(2)/U(1)$.

for a very close analysis). It amounts to use standard orbifold techniques, and we end with D-branes characterized (apart from the fermionic labels which have to do with the orientation of the brane, and the position $\hat{\mathbf{y}}$ of the brane in flat space) by $(\hat{\jmath}, \hat{m})$, corresponding respectively the $SU(2)$ spin and the $\mathbb{Z}_{2n}$ charge of a primary state of $SU(2)/U(1)$. Let's take a generic primary state $\mathcal{O}$ of the closed string theory, labeled by the left- and right-fermions numbers $(s_i, \bar{s}_i) = (a + 2\upsilon_i, \bar{a} + 2\bar{\upsilon}_i)$, the $SU(2)/U(1)$ labels $(j, m, \bar{m})$, the linear dilaton imaginary momentum $-Q\tilde{\jmath}$ as well as the $U(1)$ charges of $(i\partial X, i\bar{\partial} X)$

$$(M, \bar{M}) = (2m+ns_4+4nr, -2\bar{m}+n\bar{s}_4+4n\bar{r}) \text{ with } \quad m-s_4-4r = \bar{m}+\bar{s}_4+4\bar{r} \mod 2(n+2)\,. \tag{3.11}$$

Then the one-point function on the disc for a localized supersymmetric D-brane can be found to be, in the light-cone gauge

$$\begin{aligned}\langle \mathcal{O}^{(s_i)(\bar{s}_i)}_{p_\mu;\tilde{\jmath}M\bar{M};\,j\,m\bar{m}}(z,\bar{z})\rangle_{\hat{\jmath}\,\hat{m}\,(\hat{s}_i)\hat{\mathbf{y}}} = &|z-\bar{z}|^{-\Delta-\bar{\Delta}}\sqrt{\frac{n+2}{2}}\frac{\nu^{\frac{1}{2}-\tilde{\jmath}}}{n}\delta_{m,\bar{m}}\delta_{M,-\bar{M}}\ \delta_{s_1,\bar{s}_1}\delta_{s_2,-\bar{s}_2}\delta_{s_3,\bar{s}_3}\times\\ \times e^{i(p_2\hat{y}^2+p_3\hat{y}^3)}e^{i\frac{\pi}{2}\sum_{\ell=1}^{3}s_\ell\hat{s}_\ell}e^{-i\pi\frac{m\hat{m}}{n}}&\frac{\sin\pi\frac{(1+2j)(1+2\hat{\jmath})}{n}}{\sqrt{\sin\pi\frac{1+2j}{n}}}\frac{\Gamma(\tilde{\jmath}+\frac{M}{2(n+2)}-\frac{s_2}{2})\Gamma(\tilde{\jmath}+\frac{\bar{M}}{2(n+2)}-\frac{\bar{s}_2}{2})}{\Gamma(2\tilde{\jmath}-1)\Gamma(1+\frac{(n+2)(2\tilde{\jmath}-1)}{2n})}\,,\end{aligned} \tag{3.12}$$

with $\nu = \Gamma(1/2-1/n)/\Gamma(3/2+1/n)$. This expression has poles for values of $\tilde{\jmath}$ corresponding to discrete representations of the $\mathrm{SL}(2,\mathbb{R})$ algebra. In particular the poles associated to the operators (2.10) can be understood as the couplings between the vector multiplets and the dyon hypermultiplets dual to the D-branes in the gauge theory effective action. One can see that there is a one-to-one correspondence, see figure 1, between the geometry of the D-brane in the coset $SU(2)/U(1)$ — whose data $(\hat{\jmath}, \hat{m})$ gives the endpoints of the A-type D-brane among the n special points on the boundary of the disc [41][6] — and cycles of the Seiberg-Witten curve whose periods give the masses of the corresponding dyons in the gauge theory (see also [42]). For the particular deformation of the AD singularity preserving the $\mathbb{Z}_n$ symmetry that we consider, any homology cycle encircling two branch points of the curve will give a stable BPS dyon [43]. In the classical construction of the gauge theory with D4-branes stretched between NS5-branes, they are D2-branes with a disc topology ending on both NS5-branes and two D4-branes. In the strongly coupled quantum regime

[6]To be precise, the D-brane is stretched between the angles $\pi(\hat{m}-2\hat{\jmath}-1)/n$ and $\pi(\hat{m}+2\hat{\jmath}+1)/n$.

that is captured by the non-critical string there is no sharp bulk geometrical picture of those D-branes.

Using this one-point function one can first compute the annulus amplitude in the closed string channel, which is a sesquilinear form on the coefficients of the boundary states, with the insertion of the character for each corresponding representation propagating between the boundaries. Upon a modular transformation to the open string channel, using the modular transformation formulas given in the appendix, one obtains the following one-loop amplitude for the open strings stretched between any two of those D-branes:

$$Z_{\text{OPEN}} = \frac{q^{(\frac{\hat{\mathbf{y}}'-\hat{\mathbf{y}}}{2\pi})^2}}{\eta^2} \sum_{\{\upsilon_i\}\in(\mathbb{Z}_2)^3} \frac{1}{2} \sum_{a,b\in\mathbb{Z}_2} (-)^{a+b(1+\upsilon_1+\upsilon_2+\upsilon_3+m)} \frac{\Theta_{a+2\upsilon_1+\hat{s}_1-\hat{s}'_1,2}}{\eta} \times$$
$$\times \sum_{2j=0}^{n-2} \sum_{m\in\mathbb{Z}_{2n}} N^{j}_{\hat{j}\hat{j}'} C^{j\,(a+2\upsilon_3+\hat{s}_3-\hat{s}'_3)}_{2m+a+\hat{m}-\hat{m}'}\; Ch_{\mathbb{I}}^{(a+2\upsilon_2+\hat{s}_2-\hat{s}'_2)}(m)\,. \qquad (3.13)$$

First, local worldsheet supersymmetry imposes the condition $\hat{s}'_\ell - \hat{s}_\ell = 0 \mod 2,\ \forall \ell$. Then this sector of open strings will be spacetime-supersymmetric provided that

$$\sum_{\ell=1}^{3} (\hat{s}'_\ell - \hat{s}_\ell) + \frac{2(\hat{m}'-\hat{m})}{n} = 0 \mod 4\,. \qquad (3.14)$$

One can show that the symplectic form on the charge vectors of any pair of BPS dyons is given by the open string Witten index for strings stretched between the corresponding D-branes [38].

Open strings with both ends on any on these D-branes will always give a supersymmetric spectrum. The massless open string modes correspond to the dimensional reduction of a four-dimensional $\mathcal{N}=1$ gauge multiplet, for example in the NS sector we have the bosonic states

$$(\xi^2 \pm i\xi^3)|0\rangle_{\text{NS}} \otimes |j=0, m=0, s=0\rangle_{\text{SU(2)/U(1)}} \otimes |r=0\rangle_{\text{SL(2,R)/U(1)}}\,. \qquad (3.15)$$

Below the string scale we have also "light" multiplets of mass $\alpha' m^2 = {}^{j(j+1)}/_{n}$ for all the non-zero values of j allowed by the fusion rules $N^{j}_{\hat{j}\hat{j}'}$. Thus the dynamics of the massless degrees of freedom for N coincident D-branes is the quantum mechanics of a supersymmetric gauged matrix model. Upon T-duality in the flat space-like directions $x^{1,2,3}$, one can construct pure $\mathcal{N}=1$ SYM in four dimensions, generalizing the conifold case studied in [44, 45].

The mass of the D-branes can be obtained from the one-point function of a graviton vertex operator, coming from the continuous representations of $\mathrm{SL}(2,\mathbb{R})$ and the identity representation of $SU(2)$:

$$h^{\mu\nu} = e^{-\phi-\tilde{\phi}} e^{ip_\mu X^\mu} \psi^{(\mu}\tilde{\psi}^{\nu)} e^{-(\frac{Q}{2}+iQ\,P)\rho} V^{(0,0)}_{0\,0\,0}\,. \qquad (3.16)$$

To compute the massless tadpole associated to the graviton, we need to compute the overlap between the boundary state and the graviton closed string state with the insertion of a closed string propagator:[7]

$$
\begin{aligned}
H^{\mu\nu}(p_\mu,\,P) &= \langle h^{\mu\nu}|\frac{\alpha'}{4\pi}\int_{|z|\leqslant 1}\frac{\mathrm{d}^2 z}{z\bar{z}}\; z^{L_0-\frac{1}{2}}\;\bar{z}^{\bar{L}_0-1/2}\;|B\rangle_{\hat{\jmath}} \\
&= \frac{1}{p_\mu p^\mu+\frac{2}{\alpha'}Q^2(P^2+1/4)}\;\langle h^{\mu\nu}\rangle^{\text{DISC}}_{\hat{\jmath}}\,.
\end{aligned}
\tag{3.17}
$$

We have suppressed the extra label $\hat{m}$ of the brane since it corresponds to the position of a D-brane of a given mass and doesn't change the result. Then after a Fourier transform to position space we would obtain that the mass of the D-brane scales like

$$
m_{\text{D}}\sim\frac{1}{\sqrt{\alpha'}\,g_{\text{EFF}}}\;\sin\frac{\pi(1+2\hat{\jmath})}{n}\,,
\tag{3.18}
$$

in terms of the effective string coupling of the cigar (2.25) weighting this disc amplitude. This computation is sensitive to the proper normalization of the fields, and it would be quite difficult to give a precise definition of the ADM mass in this strongly curved background. The normalization that we would find depends on n, but since we are not interested in the large n limit in this paper it won't affect the discussion. However the ratio of masses of various dyons are not sensitive to this, and are precise predictions of the string dual of the gauge theory.

Masses of hypermultiplets and the cigar / $\mathcal{N}=2$ Liouville duality We would like now to compute in the gauge theory the masses of the light hypermultiplets corresponding to the localized D-branes. As discussed in the previous section the gauge theory is near the superconformal fixed point in the moduli space, for which the SW curve degenerates to a "small" torus of genus $\lfloor\frac{n-1}{2}\rfloor$ described by the curve

$$
y^2\simeq\pm\Lambda^n(x^n-\delta v)\,.
\tag{3.19}
$$

Then the branch points of the Seiberg-Witten curve are distributed evenly as $x_\ell=(\delta v)^{1/n}\times e^{2i\pi\ell/n}$, and the masses of the dyons that will wrap cycles of this Riemann surface will be of the same order for finite n. The exact expression for the masses will be given by integrating over these cycles the Seiberg-Witten one-form

$$
\lambda=\frac{1}{2i\pi}\frac{\partial P_n(x)}{\partial x}\frac{x\,\mathrm{d}x}{y}\,,
\tag{3.20}
$$

around any pair of branch points. If we choose the parameterization $\delta v=2\varepsilon^n$, and perform the rescaling $x=\varepsilon z$, the SW differential is given in the vicinity of the AD point by

$$
\lambda\simeq\frac{\varepsilon^{n/2+1}}{\Lambda^{n/2}}\;\frac{n}{2i\pi}\frac{z^n\,\mathrm{d}z}{\sqrt{z^n-2}}\,.
\tag{3.21}
$$

Such that all the masses of the BPS dyons near the AD point be given by integrating the Seiberg-Witten one-form over the corresponding one-cycles of the torus (rescaled by ε)

$$
m^{(i)}_{\text{DYON}}\simeq\frac{|\varepsilon|^{n/2+1}}{\Lambda^{n/2}}\left|\sqrt{2}\,\frac{n}{2i\pi}\int_{C_{(i)}}\frac{z^n\,\mathrm{d}z}{\sqrt{z^n-2}}\right|\,.
\tag{3.22}
$$

[7]For a similar computation for the conifold see [45]

The precise expression of those masses are given in terms of the multiple hypergeometric functions [46] but is not necessary for our purposes. Since, for finite n, all these dimensionless integrals are of the same order (and don't depend neither on Λ nor on ε) we get the scaling relation for the dyons masses m_{DYON} in terms of δv as:

$$m^2_{\text{DYON}} \sim (\delta v)^{\frac{n+2}{n}} \,. \tag{3.23}$$

It is very interesting that this non-trivial scaling relation can be found from worldsheet non-perturbative effects in the dual string theory. On the one hand, according to the holographic dictionary discussed above, δv, the vacuum expectation value of $s_n(\phi) = (-)^n \det(\phi)$ away from the critical value at fixed point, corresponds to the coupling constant of the $\mathcal{N}=2$ Liouville potential (2.20). On the other hand, the coefficient of cigar perturbation (2.21) is related to the effective string coupling constant. The coupling constant in type IIA superstrings is quite generically related to the masses of the localized supersymmetric D-branes, corresponding to the non-perturbative BPS states of the theory, as $\alpha' m_d^2 \sim 1/g^2_{\text{EFF}}$. In our particular example, the masses are given by (3.18).[8] Putting everything together, the scaling relation for the dyons masses predicted by the gauge theory, see eq. (3.23), gives a scaling relation between the coefficient of the geometric cigar and $\mathcal{N}=2$ Liouville perturbations on the worldsheet.

The scaling relation that we obtain matches exactly the worldsheet prediction from the duality between the supersymmetric cigar $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ at level k and $\mathcal{N}=2$ Liouville, see [48]:

$$\mu_{\text{LIOUVILLE}} = -\frac{k}{2\pi}\left(\frac{\Gamma(1+1/k)}{\Gamma(1-1/k)}\, k\pi\, \mu_{\text{CIGAR}}\right)^{k/2} , \tag{3.24}$$

with $k = 2n/n+2$ in the present case. The essential meaning of this relation is that the cigar CFT, i.e. the super-coset theory $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$, receives non-perturbative worldsheet corrections in the form of a winding condensate, the $\mathcal{N}=2$ Liouville interaction, with a coefficient given by (3.24) (the exponent itself follows from KPZ scaling [47]). Both type of operators are needed as screening charges for perturbative computations (in the worldsheet sense) of the CFT correlators. It is very interesting to see that these non-perturbative effects are mapped through the holographic duality to the exact Seiberg-Witten solution of the gauge theory, i.e. the masses of the dyons as functions of the gauge-invariant coordinates on the moduli space of the Coulomb branch. We expect that the precise coefficient appearing in this relation could be matched to the gauge theory prediction if we knew how to compute the normalization of the D-brane masses properly. We expect also that the ratio of the masses for dyons corresponding to cycles of different lengths, using (3.22), reproduce the results from the string theory dual using (3.18) since this quantities are independent of the double scaling parameter.

[8] As we have several masses in the problem, corresponding to the various D-branes in the coset $SU(2)/U(1)$, we identify the effective coupling constant with the inverse of the common mass for the elementary cycles (giving the dyons of lowest mass).

JHEP04(2006)029

4. Non-critical superstrings with $\mathcal{N}=1$ supersymmetry

Starting from the previous construction of strings duals of $\mathcal{N}=2$ gauge theories, it is possible to construct new four-dimensional non-critical superstring theories with only one spacetime supersymmetry, that will be solvable string duals of $\mathcal{N}=1$ gauge theories and their little string theory UV completion. These theories will also be at non-trivial superconformal fixed points, or in the neighborhood of those, and the spectrum of massless string modes will give the scaling dimensions of chiral operators in the superconformal field theory.

4.1 Asymmetric orbifolds of four-dimensional non-critical superstrings

To construct these new non-critical strings we will perform an asymmetric orbifold of the worldsheet CFT, quite similar to a lens space, acting only on the left-movers of the worldsheet. In the context of six-dimensional LST[9], one can replace the $SU(2)_k \sim [SU(2)/U(1)|_k\ U(1)_k]/\mathbb{Z}_k$ super-WZW model by a lens space, i.e. a $\mathbb{Z}_p \backslash SU(2)_k$ chiral orbifold [51] acting on the $SU(2)$ group elements as $g \to \exp \frac{2i\pi}{p}\sigma_3\, g$. The six-dimensional holography associated to this lens space has been studied in [53].

In the four dimensional little string theory studied in this paper we can construct a similar asymmetric orbifold structure, this time acting on $[SU(2)/U(1)|_n \times U(1)_{2n(n+2)}]$, provided that we choose the level of the coset as

$$n = pp' \quad \text{with} \qquad p,\, p' \in \mathbb{Z}\,. \tag{4.1}$$

This restriction will be understood later from the gauge theory point of view. The partition function (2.8) of the original model for the continuous representations is replaced by the asymmetric $\mathbb{Z}_p$ orbifold partition function

$$\begin{aligned} Z_{\text{CONT}}(\tau,\bar\tau) = \frac{1}{4\pi^2\alpha'\tau_2}\frac{1}{\eta^2\bar\eta^2}\,\frac{1}{4}\sum_{a,b,\bar a,\bar b\in\mathbb{Z}_2}\ \sum_{\{\upsilon_\ell\},\{\bar\upsilon_\ell\}\in(\mathbb{Z}_2)^4} (-)^{a+\bar a+b(1+\sum_i \upsilon_i)+\bar b(1+\sum_i\bar\upsilon_i)+a\bar b} \times \\ \times\frac{\Theta_{a+2\upsilon_1,2}\Theta_{\bar a+2\bar\upsilon_1,2}}{\eta\bar\eta}\int_0^\infty \mathrm{d}P\sum_{2j=0}^{n-2}\ \sum_{m,\bar m\in\mathbb{Z}_{2n}}\ \sum_{r\in\mathbb{Z}_{n+2}} \delta_{m-a-2\upsilon_4-4r,\bar m+\bar a+2\bar\upsilon_4+4\bar r \mod 2(n+2)} \times \\ \times\, Ch_c^{(a+2\upsilon_2)}(P,2m+n(a+2\upsilon_4)+4nr)\bar{C}h_c^{(\bar a+2\bar\upsilon_2)}(P,-2\bar m+n(a+2\bar\upsilon_4)+4n\bar r) \times \\ \times\sum_{\gamma\in\mathbb{Z}_p}\delta_{m-p'\gamma,0 \mod p}\ C^{j\,(a+2\upsilon_3)}_{m-2p'\gamma}\ \bar C^{j\,(\bar a+2\bar\upsilon_3)}_{\bar m}\,, \end{aligned} \tag{4.2}$$

that is modular invariant as can be checked explicitly. The states invariant under the $\mathbb{Z}_p$ orbifold action are selected by the Kronecker delta in the last line and the twisted sectors, requested by modular invariance, are labeled by the integer γ taking values in $\mathbb{Z}_p$. In particular the spacetime supercharges from the left-movers in the 3+1 LST, given by

[9] Whose holographic dual is the CHS background [52] $\mathbb{R}^{5,1}\times\mathbb{R}_Q\times SU(2)$.

eq. (2.9), will be projected out because they have $m = \pm 1$. On the other hand, since the orbifold doesn't act on the right, the spacetime supercharges from the right-movers are preserved. We end up with a non-critical string theory with only $\mathcal{N} = 1$ supersymmetry in four dimensions, albeit without Ramond-Ramond fluxes. From this partition function we can read the spectrum of string states that are dual to off-shell operators in the dual theory. Our aim will be to show that this non-critical string theory provides an example of holographic dual of $\mathcal{N} = 1$ gauge theories that is exactly solvable. This theory will also flow to a non-trivial superconformal fixed point, if there is no potential along the linear dilaton direction.

4.2 Chiral spectrum and double scaling limit

Let's first discuss the spacetime chiral operators that we can read from this partition function. The $U(1)_{\tilde{R}}$ symmetry that appears in the $SU(2,2|1)$ superconformal algebra gives the scaling dimension (at the superconformal fixed point) of the chiral/antichiral primary operators as

$$\Delta \; = \; \frac{3}{2}|\tilde{R}| \; . \tag{4.3}$$

This $U(1)_{\tilde{R}}$ is given in terms of the $\mathcal{N} = 2$ $U(1)_{\rm R}$ charge R, see eq. (2.14), and the $U(1) \subset SU(2)_{\rm R}$ charge $m_{\rm R}$, which are both R-symmetries of the $\mathcal{N} = 1$ theory, by the following linear combination

$$\tilde{R} = \frac{R - 4m_{\rm R}}{3} = \frac{2}{3Q}\oint \left[\partial X - \bar{\partial} X - 2(\partial X + \bar{\partial} X)\right] \; , \tag{4.4}$$

such that the surviving supercharges have charge ± 1 (they have only right-moving momenta from the worldsheet point of view).[10] Let us look first at the untwisted NS-NS sector. We can build operators similar to those $\mathcal{N} = 2$ models, see eq. (2.10), but they have to be invariant under the orbifold action:

$$\mathcal{V}^{\rm U}_{\frac{pN}{2}-1} = e^{-\varphi-\tilde{\varphi}} e^{-Q\tilde{\jmath}\rho} e^{i\sqrt{\frac{p}{p'(pp'+2)}}N(X_L - X_R)} V^{(2,2)}_{\frac{pN}{2}-1,\, \frac{pN}{2},\, \frac{pN}{2}} \tag{4.5}$$

constructed from $SU(2)$ representations of spin $j = pN/2 - 1$. These operators are dual to chiral operators in the $\mathcal{N} = 1$ gauge theory which follows from the fact that they were the bottom components of the vector multiplet in the $\mathcal{N} = 2$ theory that was orbifoldized. Using the definition (4.4) of the R-charge in spacetime, we obtain the following spectrum of scaling dimensions for these operators at the superconformal fixed point:

$$\Delta_{\rm U} = \frac{4pN}{pp'+2} \quad \text{for} \qquad N > \frac{p'}{2} + \frac{1}{p} \, . \tag{4.6}$$

The scaling dimension of these chiral operators is the same as those of the $\mathcal{N} = 2$ chiral operators of the parent theory they are coming from. This could be understood e.g. in the M-theory AdS description of the superconformal theory discussed above, since of course

[10] If we would have done instead the orbifold on the right-moving worldsheet CFT, preserving the left-moving spacetime supercharges, the correct definition would have been $\tilde{R} = \frac{R+4m_{\rm R}}{3}$.

JHEP04(2006)029

the asymptotic behavior of the bulk fields in AdS_5 coupling to these chiral operators won't change with the orbifold; it will simply project out eigenfunctions for the Laplacian on X^6 that are not invariant under the orbifold action.

The orbifold theory contains also twisted sectors, labeled by $\gamma \in \mathbb{Z}_p$. For each such sector we can construct chiral operators in spacetime as follows:

$$\mathcal{V}^{\mathrm{T}}_{\frac{p\gamma}{2}-1} = e^{-\varphi-\tilde{\varphi}} e^{-Q\tilde{\jmath}\rho} e^{i\sqrt{\frac{p'}{p(pp'+2)}}\gamma(X_L - X_R)} V^{(2,2)}_{\frac{p'\gamma}{2}-1, -\frac{p'\gamma}{2}, \frac{p'\gamma}{2}} \tag{4.7}$$

These states violate the integrality condition of the left worldsheet $\mathcal{N}=2$ R-charge because the $\mathbb{Z}_{2n}$ charge for the left-moving sector of $SU(2)/U(1)$ has a negative sign, breaking the spacetime supersymmetry associated to the $\mathcal{N}=2$ spectral flow. However the spacetime supersymmetry associated with the right-movers is preserved. Again, eq. (4.4) gives the R-charge and thus the scaling dimension in spacetime of those operators:

$$\Delta_{\mathrm{T}} = \frac{4p'\gamma}{pp'+2} \quad \text{for} \qquad \gamma > \frac{p}{2} + \frac{1}{p'} \,. \tag{4.8}$$

It is quite interesting to note that the T-duality between the $\mathbb{Z}_p$ and the $\mathbb{Z}_{p'}$ orbifolds, similar to the T-duality between the lens spaces ${}_{\mathbb{Z}_p}\backslash SU(2)_{pp'} \overset{\mathrm{T}}{\longleftrightarrow} {}_{\mathbb{Z}_{p'}}\backslash SU(2)_{pp'}$, translates into a symmetry in the spectrum for the gauge theory in the neighborhood of the superconformal fixed point. Indeed the chiral operators from the untwisted sector (4.6) and from the twisted sectors (4.8) are exchanged under $p \leftrightarrow p'$, giving overall the same spectrum of scaling dimensions in the four-dimensional superconformal field theory.

As in the $\mathcal{N}=2$ models, it is possible to regularize the strong coupling region of the linear dilaton by taking a double scaling limit, i.e. by replacing the $\mathbb{R}_Q \times U(1)$ factor by a super-coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$. The structure of the coset CFT is not affected by the orbifold. In particular we see that the $\mathcal{N}=2$ Liouville operator is still in the spectrum, and can be viewed either as an untwisted sector operator (4.6) with $N=p'$ or a twisted sector one (4.8) with $\gamma = p$. Then the one-loop vacuum amplitude will also contain a contribution for discrete representations, that will be similar to (2.26) upon replacing

$$C^{j\,(a+2\upsilon_3)}_{m} \quad \to \quad \sum_{\gamma \in \mathbb{Z}_p} \delta_{m-p'\gamma,0 \mod p} \; C^{j\,(a+2\upsilon_3)}_{m-2p'\gamma} \,, \tag{4.9}$$

as for the continuous representations (4.2). The worldsheet chiral operators belonging to this discrete spectrum will be in one-to-one correspondence with relevant deformations of the $\mathcal{N}=1$ superconformal field theory in spacetime, as in the $\mathcal{N}=2$ models.

4.3 Brane setup for the gauge theory

In order to find the gauge theory duals of these $\mathcal{N}=1$ non-critical strings in the semi-classical regime, we can use the NS5/D4 construction in type IIA superstrings of the $\mathcal{N}=2$ SYM theories discussed in section 3, and show how the asymmetric orbifold of the non-critical string is implemented at the level of this brane setup. We consider again a pair of

NS5-branes of worldvolume $x^{0,1,2,3,4,5}$ located at $x^6 = 0$, $x^6 = L$ and $x^7 = x^8 = x^9 = 0$. They preserve the supercharges corresponding to spinors solutions of

$$\eta_{\substack{\mathrm{L}\\\mathrm{R}}} = \pm\gamma^0\gamma^1\gamma^2\gamma^3\gamma^4\gamma^5 \; \eta_{\substack{\mathrm{L}\\\mathrm{R}}} \,, \tag{4.10}$$

where the left and right supercharges η_{L} and η_{R} have opposite chiralities. We stretch between the fivebranes pp' D4-branes, of worldvolume $x^{0,1,2,3,6}$, distributed in the $x^{4,5}$ plane. They preserve the following supercharges

$$\eta_{\mathrm{L}} = \gamma^0\gamma^1\gamma^2\gamma^3\gamma^6 \; \eta_{\mathrm{R}} \,. \tag{4.11}$$

On top of this brane configuration we can add a $\mathbb{C}^2/\mathbb{Z}_p$ orbifold, acting in the planes $x = x^4 + ix^5$ and $h = x^8 + ix^9$ as follows

$$x \to e^{-\frac{2i\pi}{p}} \, x \quad , \qquad h \to e^{\frac{2i\pi}{p}} \, h \,, \tag{4.12}$$

which is a symmetry of the brane configuration, provided that the D4-branes are distributed in a $\mathbb{Z}_p$-symmetric fashion in the $(4,5)$ plane; in other words we consider p' D4-branes in the fundamental domain of the orbifold. This orbifold preserves the left and right supercharges that satisfy

$$\eta_{\substack{\mathrm{L}\\\mathrm{R}}} = \gamma^4\gamma^5\gamma^8\gamma^9 \; \eta_{\substack{\mathrm{L}\\\mathrm{R}}} \,, \tag{4.13}$$

leading overall to $\mathcal{N} = 1$ in four dimensions. Such models have already been considered in [16], however the construction of the dual non-critical string is new.

The identification of this brane construction – that describes the gauge theory dual in the semi-classical regime — with our $\mathcal{N} = 1$ four-dimensional non-critical string is as follows. The action of the orbifold on the $x = x^4 + ix^5$-plane can be seen from the dual formulation of the four-dimensional non-critical string as a singular Calabi-Yau three-fold compactification in type IIB, see e.g. [2]. One can relate the $\mathcal{N} = 2$ non-critical strings of the sort (2.3) to the decoupling limit of the singular CY$_3$ given by the hypersurface $x^n + y^2 + uv = 0$ embedded in $\mathbb{C}^4$.[11] To make the connection more obvious we consider another representation of the worldsheet CFT as the infrared limit of a Landau-Ginzburg model with the potential $X^{pp'} + Y^2 + UV + \mu Z^{-\frac{2n}{n+2}}$, the last factor corresponding to the $\mathcal{N} = 2$ Liouville/cigar theory if the singularity is resolved. The elements of the chiral ring of this theory are the monomials X^{2j+2}, dressed appropriately by the $\mathcal{N} = 2$ Liouville system as we explained above in a different language. The orbifold that we consider in the non-critical string (4.2) leaves invariant only the elements of the chiral ring of the form X^{pN}, $N \in \mathbb{N}$. It is then identified as a $\mathbb{Z}_p$ rotation in the complex $x = x^4 + ix^5$ plane, in the brane construction described above. The action in the h-plane, see eq.(4.12), is dictated by supersymmetry (because a $\mathbb{C}/\mathbb{Z}_p$ orbifold alone breaks supersymmetry). The symmetries of the brane construction allows only this orbifold to act on a $U(1)$ subgroup of the $SU(2)_R$ symmetry corresponding to rotations in the $x^{7,8,9}$ overall transverse directions, preserving $\mathcal{N} = 1$ supersymmetry in four dimensions.

[11] This description is indeed related by a T-duality to the description in terms of an NS5-brane wrapping the curve $x^n + y^2 = 0$ discussed above.

Let us analyze briefly the gauge theory corresponding to this configuration. We start with pp' D4-branes having on their worldvolume an $\mathcal{N}=2$ $U(pp')$ gauge multiplet in 4+1 dimensions. It contains an $\mathcal{N}=1$ gauge multiplet, whose scalar component corresponds to the fluctuations along x^7, and a hypermultiplet, whose four scalar components correspond to fluctuations along $x^{4,5,8,9}$. The action of the orbifold on the pp' D4-branes gives an $U(p')^p$ quiver gauge theory with $\mathcal{N}=1$ in five dimensions. Now we suspend these D4-branes between two NS5-branes along the x^6 direction; since the NS5-branes are very heavy we consider only the degrees of freedom living on the D4-branes. Because the D4-branes end on NS5-branes, the boundary conditions at their endpoints will remove part of the massless fields and break half of the supersymmetry on their worldvolume, as suggested in [54, 35]. These boundary conditions will indeed set to zero the fluctuations along $x^{6,7,8,9}$. We end up at low energies (compared to the inverse of the distance between the fivebranes) with $\mathcal{N}=1$ gauge multiplets in four dimensions and bifundamental chiral multiplets. As for the $\mathcal{N}=2$ models the diagonal $U(1)$ factor is frozen. The other $U(1)$'s are also all anomalous (for $p>2$), leaving an $SU(p')^p$ quiver [16]. For a generic vacua this gauge symmetry is broken to $U(1)^{p'-1}$, thus we are in an Abelian Coulomb phase. We will give below a better derivation of the open string theory living on the D4-branes.

4.4 Boundary worldsheet CFT construction

The construction that we outlined in the previous section gave the correct field content of the gauge theory on the D4-branes, however it is somehow heuristic since the analysis of the D4-branes ending on NS5-branes was only qualitative. It is possible to give a precise description of this configuration, including the backreaction of the NS5-branes, along the lines of [37] where the $\mathcal{N}=2$ case was studied. We analyze the system from the worldsheet point of view, starting from the worldsheet CFT describing the background for the near-horizon geometry of two separated NS5-branes in type IIA. This is a good approximation since the NS5-branes are considered in these constructions are very heavy, non-fluctuating objects. Our aim is to derive more rigorously the construction of the $\mathcal{N}=1$ gauge theory explained above, by constructing the D4-brane boundary state in the presence of the fivebranes and of the orbifold.

The worldsheet CFT for two flat parallel fivebranes corresponds to the string background $\mathbb{R}^{5,1}\times\mathbb{R}_{Q=1}\times SU(2)_2$, i.e. eight-dimensional non-critical superstrings, which is a particular case of the CHS background. The two fivebranes can be separated in the transverse space if we replace $\mathbb{R}_Q\times U(1)_2$ with the super-coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ at level $k=2$. The supersymmetric SU(2) WZW model at level two contains only the fermionic $SU(2)_2$ affine currents (since the bosonic algebra is at level zero), constructed with three fermions ψ^3, $\psi^\pm=(\psi^1\pm i\psi^2)/\sqrt{2}$:

$$J^3=\psi^+\psi^- \quad , \qquad J^\pm=i\sqrt{2}\,\psi^\pm\psi^3\,. \tag{4.14}$$

In terms of those, and the remaining fermions $\xi^{2,3,4,5,\rho}$ of $\mathbb{R}^4\times\mathbb{R}_Q$ the generators of the worldsheet $\mathcal{N}=2$ superconformal algebra – which is actually enhanced to $\mathcal{N}=4$ [55]–

read

$$
\begin{aligned}
-2iG^{\pm} &= (\partial X^2 \mp i\partial X^3)(\xi^2 \pm i\xi^3) + (\partial X^4 \mp i\partial X^5)(\xi^4 \pm i\xi^5) \\
&\quad + \left[\partial\rho \pm i(\psi^1\psi^2 - \xi^\rho\psi^3)\right](\xi^\rho \pm i\psi^3) \\
-iJ_R &= \xi^2\xi^3 + \xi^4\xi^5 + \xi^\rho\psi^3 + \psi^1\psi^2
\end{aligned} \tag{4.15}
$$

The action of the orbifold (4.12) in the transverse space translates into the following action on the fermionic $SU(2)$ left- and right-moving affine currents and thus on the free worldsheet fermions[12]

$$
\begin{array}{lcl}
J_3 \to J_3\ ,\quad J^{\pm} \to e^{\pm\frac{2i\pi}{p}} J^{\pm} & \Longrightarrow & \psi^3 \to \psi^3\ ,\quad \psi^{\pm} \to e^{\pm\frac{2i\pi}{p}}\psi^{\pm} \\
\tilde{J}_3 \to \tilde{J}_3\ ,\quad \tilde{J}^{\pm} \to e^{\pm\frac{2i\pi}{p}} \tilde{J}^{\pm} & \Longrightarrow & \tilde{\psi}^3 \to \bar{\psi}^3\ ,\quad \tilde{\psi}^{\pm} \to e^{\pm\frac{2i\pi}{p}}\tilde{\psi}^{\pm}\ ,
\end{array}
$$

together with an action of the orbifold on the free fields corresponding to the $(4,5)$ plane as an ordinary $\mathbb{C}/\mathbb{Z}_p$ orbifold. This preserves of course the $\mathcal{N}=2$ algebra on the worldsheet but keeps only half of the spacetime supercharges. Next we consider the double scaling limit of this little string theory by separating the NS5-branes in the $(6,7)$ plane, which is not affected by the orbifold. The type IIA partition function of the orbifold theory, for the continuous representations of $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$, is given by

$$
Z = \frac{1}{4\pi^2\alpha'\tau_2}\frac{1}{\eta^2\bar\eta^2}\frac{1}{4p}\sum_{a,b,\bar a,\bar b}(-)^{a+b+\bar a+\bar b+\bar a\bar b}\frac{\vartheta\left[{a \atop b}\right]}{\eta}\frac{\bar\vartheta\left[{\bar a \atop \bar b}\right]}{\bar\eta}\int \mathrm{d}P\,\frac{(q\bar q)^{\frac{P^2}{2}}}{\eta\bar\eta}\frac{\vartheta\left[{a \atop b}\right]}{\eta}\frac{\bar\vartheta\left[{\bar a \atop \bar b}\right]}{\bar\eta}\ \times
$$
$$
\left[\frac{1}{4\pi^2\alpha'\tau_2|\eta|^4}\frac{\vartheta\left[{a \atop b}\right]^2}{\eta^2}\frac{\bar\vartheta\left[{\bar a \atop \bar b}\right]^2}{\bar\eta^2} + \sum_{\gamma,\delta\in\mathbb{Z}_p\neq(0,0)}\frac{|\eta|^2}{\left|\vartheta\left[{1+2\gamma/p \atop 1+2\delta/p}\right]\right|^2}\frac{\vartheta\left[{a-2\gamma/p \atop b-2\delta/p}\right]}{\eta}\frac{\bar\vartheta\left[{\bar a-2\gamma/p \atop \bar b-2\delta/p}\right]}{\bar\eta}\frac{\vartheta\left[{a+2\gamma/p \atop b+2\delta/p}\right]}{\eta}\frac{\bar\vartheta\left[{\bar a+2\gamma/p \atop \bar b+2\delta/p}\right]}{\bar\eta}\right] \tag{4.16}
$$

which has two supercharges in four dimensions, one from the left-movers and one from the right movers. The last $\vartheta\left[{\cdot \atop \cdot}\right]\bar\vartheta\left[{\cdot \atop \cdot}\right]/\eta\bar\eta$ factor corresponds to the fermions $(\psi^{\pm},\tilde\psi^{\pm})$ of the fermionic $SU(2)$ that we discussed before, or equivalently to the compact boson of the cigar which is asymptotically at the fermionic radius, for which the twisted sectors of the $\mathbb{Z}_p$ orbifold translate into fractional windings γ/p. The relative sign of the orbifold action on the left- and right- moving sector is such that the $\mathcal{N}=2$ Liouville operator is not projected out. From the geometrical point of view it means that the orbifold has to act in the (x^8,x^9) plane and not in the (x^6,x^7) one.

Now we can add a D4-brane stretched between the two NS5-branes, localized in the plane $(4,5)$. Without the orbifold, the one-point function for the D4-brane in type IIA reads [37, 38], (in the light-cone gauge)

$$
\begin{aligned}
\langle\, V^{(s_i)\ (\bar s_i)}_{\tilde\jmath,\mathbf{p}}\,\rangle_{\hat s_i,\hat{\mathbf{y}}} &= |z-\bar z|^{-\Delta-\bar\Delta}\ 2^{-\frac12-\tilde\jmath}\ \delta_{s_1,-\bar s_1}\delta_{s_2,-\bar s_2}\delta_{s_3,-\bar s_3}\delta_{s_4,-\bar s_4}\ \delta^{(2)}(\mathbf{p})\ \times \\
&\quad\times\ e^{i(p_4\hat y^4+p_5\hat y^5)}e^{i\frac{\pi}{2}\sum_i s_i\hat s_i}\ \frac{\Gamma\left(\tilde\jmath+\frac{s_4-s_3}{2}\right)\Gamma\left(\tilde\jmath+\frac{\bar s_4-\bar s_3}{2}\right)}{\Gamma(2\tilde\jmath-1)\Gamma(\frac12+\tilde\jmath)}\ ,
\end{aligned} \tag{4.17}
$$

[12]This orbifold is different from the lens space since it acts non-chirally.

for a closed string vertex operator with SL$(2,\mathbb{R})$ spin $\tilde{j}$ (i.e. of the form $V \sim e^{-Q\tilde{j}\phi}$) fermionic R-charges $(s_i, \bar{s}_i)$ for $i = 1, \ldots, 3$ (the last ones being the fermions of the cigar). The charges $(s_4, \bar{s}_4)$ correspond to the left and right chiral momenta of the compact boson in the cigar SL$(2,\mathbb{R})$/U(1). This one-point function gives the following open string annulus amplitude for open strings with both ends on the same D-brane:

$$Z_{\text{OPEN}}(\tau) = \tag{4.18}$$
$$= \frac{1}{-i\tau 8\pi^2\alpha'\eta^4} \frac{1}{2} \sum_{a,b=0}^{1} \sum_{\{\upsilon_\ell\}\in(\mathbb{Z}_2)^4} (-)^{a+b(1+\sum_\ell \upsilon_\ell)} \frac{\Theta_{a+2\upsilon_1,2}\Theta_{a+2\upsilon_2,2}}{\eta^2} Ch_{\mathbb{I}}^{(a+2\upsilon_3)}(\upsilon_4;\tau)\ .$$

The massless spectrum corresponds to the dimensional reduction of a six-dimensional $\mathcal{N}=1$ gauge multiplet to four dimensions. For $n = pp'$ coincident D4-branes, one obtains an $\mathcal{N}=2$ $U(n)$ gauge multiplet. It contains, from the NS sector, a four-dimensional gauge field, given by

$$\lambda_{\text{G}}\ A_\mu \psi^\mu_{-1/2}\ |0\rangle_{\text{NS}}\ \otimes\ |p^\mu\rangle\ \otimes\ |r=0\rangle_{\text{SL}(2,\text{R})/\text{U}(1)}\ , \tag{4.19}$$

and a complex scalar

$$\lambda_{\text{C}}\ (\xi^4 \pm i\xi^5)_{-1/2}\ |0\rangle_{\text{NS}}\ \otimes\ |p^\mu\rangle\ \otimes\ |r=0\rangle_{\text{SL}(2,\text{R})/\text{U}(1)}\ . \tag{4.20}$$

The $pp' \times pp'$ matrices λ_{G} and λ_{C} are associated with the Chan-Paton factors. Before the orbifold they are both arbitrary Hermitian matrices, associated to the adjoint representation of $U(pp')$. There are no other massless degrees of freedom since all the excitations along the SL$(2,\mathbb{R})$/U(1) factor are massive. Now we perform the $\mathbb{Z}_p$ orbifold of the open string theory. In particular we have to find the action of the orbifold group onto the matrices associated to the Chan-Paton factors. The solution to this problem is quite similar to the construction of D-branes transverse to an ALE space [14, 56], in the present case to an A_{p-1} singularity, since the action of the $\mathbb{Z}_p$ orbifold is Abelian. The action of the orbifold on the SL$(2,\mathbb{R})$/U(1) open string operators in the NS sector ($a=0$), will be only on the massive states of the open string theory since the vacuum $r=0$ of SL$(2,\mathbb{R})$/U(1) is invariant. One obtains then,from the action on the $x^{4,5}$ directions and on the Chan-Paton factors, an $\mathcal{N}=1$ quiver $U(p')^p$, for pp' D4-branes on the covering space of the orbifold. The excitations along $x^{4,5}$ turns into bifundamental chiral multiplets $Q_{\ell\to\ell+1}$ connecting the nodes of the affine $\hat{A}_{p-1}$ Dynkin diagram. This quiver is a *chiral* one since there is only one chiral multiplet associated to each link. Then as explained above the $U(1)$ are all anomalous and we end up with an $SU(p')^p$ gauge group.

From the construction of the gauge theory using boundary worldsheet CFT outlined above we obtain only the classical properties of the field theory. The perturbative corrections can be found by computing the bending of the NS5-branes — corresponding to the one-loop running of the coupling constant — as a closed string tadpole for the string states holographically dual to the operators $\text{Tr}(X^6 + iX^7)^r$ on the fivebranes worldvolume [37]. It would be also interesting to compute the $U(1)$ anomalies from the worldsheet boundary CFT approach; they should correspond to tadpoles in the R-R twisted sectors of the $\mathbb{Z}_p$ orbifold. To describe the full quantum theory in the strong coupling regime one needs to go to

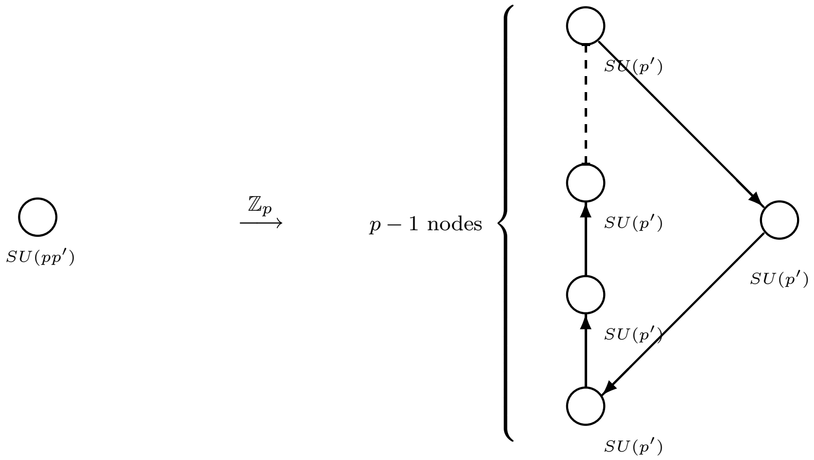

Figure 2: Quiver diagram obtained by a $\mathbb{Z}_p$ orbifold of the $SU(pp')$ theory.

the four-dimensional non-critical strings regime, for which this configuration corresponds to a single NS5-brane wrapping a singular Riemann surface embedded in $\mathbb{C} \times \mathbb{C}/\mathbb{Z}_p$.

5. Double scaling limit of $\mathcal{N} = 1$ quivers

We have demonstrated above that an asymmetric orbifold of $\mathcal{N} = 2$ non-critical strings leads to an $\mathcal{N} = 1$ string theory, holographically dual to a non-gravitational theory flowing at low energies to an $\mathcal{N} = 1$ affine $\hat{A}_{p-1}$ quiver gauge theory, in the Coulomb phase. To each node of the quiver is associated an $\mathcal{N} = 1$ vector multiplet and to each link a bifundamental $\mathcal{N} = 1$ chiral multiplet $Q_{\ell\to\ell+1}$ transforming in $(\bar{\Box}, \Box)$, see figure 2.[13] Since there is only one oriented link between adjacent nodes, i.e. chiral multiplets transforming in $(\bar{\Box}, \Box)$ and not in $(\Box, \bar{\Box})$, this quiver gauge theory is chiral. It has no superpotential, because there is no superpotential inherited from the projection of the $\mathcal{N} = 2$ pure gauge theory – in opposition with what happens for $\mathcal{N} = 1$ gauge theories coming from orbifolds of $\mathcal{N} = 4$ theories [60]. The absence of superpotential could be checked from the worldsheet boundary CFT construction of the previous section, since each term in the superpotential can be computed as open string n-point functions of the appropriate boundary fields, as was done in [61] for Gepner models. The action of the orbifold on the open string field theory, in particular on the chiral multiplets (4.20), won't generate new nonzero correlators.

From our construction it is quite clear that the linear dilaton limit of the non-critical string corresponds to a non-trivial $\mathcal{N} = 1$ superconformal field theory on the gauge theory side. This superconformal gauge theory is an Argyres-Douglas point in the Coulomb phase of the affine $\hat{A}_{p-1}$ quiver, where the gauge symmetry is generically $U(1)^{p'-1}$. As for the $\mathcal{N} = 2$ models we will also study the double scaling limit of the non-critical string dual, corresponding to the neighborhood of the superconformal fixed point.

[13] Note that this gauge theory can also be obtained by deconstruction of five-dimensional SYM [57 – 59].

5.1 Chiral ring of the quiver

In order to understand the matching between the chiral ring of the quiver and the string theory massless operators that are chiral in spacetime, we start by describing the moduli space of the gauge theory and the associated SW curve, following [15, 62]. A basis of independent gauge-invariant generators of the chiral ring (omitting for the moment the operators involving the gaugino superfield) is provided by

$$\begin{aligned} \mathfrak{L}_r &= \mathrm{Tr}(Q_{1\to 2}Q_{2\to 3}\cdots Q_{p\to 1})^r \ , \qquad & r=1,\dots,p'-1 \\ \mathfrak{B}_{\ell\to\ell+1} &= \det Q_{\ell\to\ell+1} \ , & \ell\sim\ell+p \end{aligned} \tag{5.1}$$

The first type of operators are made from the "loop" operator, i.e. the quiver-ordered product of all the link fields. It can be used to define a composite adjoint operator

$$\Xi = Q_{1\to 2}Q_{2\to 3}\cdots Q_{p\to 1} - \frac{1}{p'}\,\mathfrak{L}_1\ \mathbb{I}_{p'\times p'} \ . \tag{5.2}$$

The second type of operators are "baryons" and are the generators of a larger class of determinants of products of link fields $\mathfrak{B}_{\ell\to\ell+p} = \det(Q_{\ell\to\ell+1}\cdots Q_{\ell+p-1\to\ell+p})$. The classical constraint between those and the "elementary" baryons operators is modified quantum mechanically [63]. In particular using such a relation we can express $\mathfrak{L}_{p'} = \mathrm{Tr}(Q_{1\to 2}\cdots Q_{p\to 1})^{p'}$ in terms of the other gauge-invariant operators, as follows [64, 65]

$$\mathfrak{B}_{1\to 1} = \mathfrak{B}_{1\to 2}\times\cdots\times\mathfrak{B}_{p\to 1} + \left(\mathfrak{B}_{r\to r+1}\mathfrak{B}_{r+1\to r+2}\to\Lambda_{r+1}^{2p'}\right) \tag{5.3a}$$

$$= \frac{(-1)^{p'+1}}{p'}\ \mathfrak{L}_{p'} + \sum_{r=2}^{p'}\sum_{\{n_i\}=1}^{p'-1}\delta_{\sum n_i,p'}\frac{(-)^{r+p'}}{r!}\frac{\mathfrak{L}_{n_1}}{n_1}\cdots\frac{\mathfrak{L}_{n_r}}{n_r} \tag{5.3b}$$

where on the right-hand side of (5.3a) we mean that we must add all the possible terms obtained by replacing pairs of adjacent baryonic operators $(\mathfrak{B}_{r\to r+1}\mathfrak{B}_{r+1\to r+2})$ by $\Lambda_{r+1}^{2p'}$, corresponding to the dynamically generated scale for the $(r+1)$-th $SU(p')$ gauge group on the quiver diagram.

The properties of the Coulomb phase for $\mathcal{N}=1$ gauge theories are given, as for the $\mathcal{N}=2$ models in terms of an hyperelliptic curve [66], which allows to compute the gauge coupling τ as a function of the coordinates on the moduli space. For the $SU(p')$ quivers associated to the $\hat{A}_{p-1}$ diagram that we consider, the curve is given by

$$y^2 = \frac{1}{4}\langle\det(x-\Xi)\rangle^2 - \Lambda_1^{2p'}\cdots\Lambda_p^{2p'} \tag{5.4}$$

Using the relation (5.3) the parameter $s_{p'}(\Xi)$ of this curve can be related to the vacuum expectation values of the chiral ring operators (5.1). In our particular gauge theory derived from a string theory construction, the vacuum expectations values of all the baryonic operators are related by D-terms constraints, coming from the anomalous $U(1)$ gauge symmetries discussed in the previous section. Because of the $\mathbb{Z}_p$ symmetry of the theory we also consider all the scales Λ_ℓ to be the same. This curve have the same type of singularities as the $\mathcal{N}=2$ $SU(p')$ curve that we discussed in the previous sections. In terms of the parameters (5.1) these singularities are not points but rather hypersurfaces of the moduli space. They correspond to the same number of dyons becoming massless, although their mass cannot be determined a priori since they are non-holomorphic quantities. In particular, for $p' > 2$ the moduli space contains singularities of the Argyres-Douglas type where mutually non-local dyons become massless, leading to a strongly interacting $\mathcal{N} = 1$ superconformal field theory in the infrared. As for the $\mathcal{N} = 2$ models, the string dual describe these singularities and — when the strong coupling singularity of the linear dilaton is resolved — small perturbations of those.

Now that we have a clear picture of the moduli space of the Coulomb branch, we can find the mapping between the gauge-invariant chiral operators (5.1) of the quiver and string massless states that are chiral in spacetime. First we can identify the invariant polynomials of the composite adjoint Ξ, defined in eq. (3.1), with string NS-NS chiral primaries from the untwisted sectors:

$$\mathcal{V}^{\mathrm{U}}_{\frac{pN}{2}} = e^{-\varphi-\tilde{\varphi}} e^{-Q\tilde{\jmath}\rho} e^{i\sqrt{\frac{p}{p'(pp'+2)}}N(X_L-X_R)} V^{(2,2)}_{\frac{pN}{2}-1,\,\frac{pN}{2},\,\frac{pN}{2}} \longleftrightarrow s_N(\Xi)\ , \quad N = 0,\dots,p'-1\,. \tag{5.5}$$

Only the operators with $N > p'/2 + 1/p$ will correspond to string theory observables, and from the gauge theory point of view they satisfy the unitarity bound at the superconformal fixed point. It gives the following holographic prediction for the scaling dimensions of such chiral operators

$$\Delta[s_N(\Xi)] \;=\; \frac{4pN}{pp'+2} \tag{5.6}$$

with the same issue of mixing as discussed in the $\mathcal{N} = 2$ case, see eq. (3.6). This spectrum of dimensions is different from the $\mathcal{N} = 2$ theory corresponding to the diagonal $SU(p')$ of the quiver which is not Higgsed (but broken to $U(1)^{p'-1}$ on the Coulomb branch). This seems surprising since the two theories share the same type of SW curve. In the $\mathcal{N} = 2$ theories, the overall normalization of the R-charges for the perturbations around the singularity can be found by requiring that the Kähler potential has dimension two, which implies that the Seiberg-Witten one-form has dimension one [67]. However, for the $\mathcal{N} = 1$ theories, the Kähler potential is not given by the curve and such a constraint does not apply.

As we saw the chiral ring contains also "baryonic" operators made of determinants of link fields. They are identified with NS-NS worldsheet chiral primaries in the twisted sector of the string theory orbifold as follows

$$\mathcal{V}^{\mathrm{T}}_{\frac{p'\gamma}{2}} = e^{-\varphi-\tilde{\varphi}} e^{-Q\tilde{\jmath}\rho} e^{i\sqrt{\frac{p'}{p(pp'+2)}}\,\gamma\,(X_L-X_R)} V^{(2,2)}_{\frac{p'\gamma}{2}-1,\,-\frac{p'\gamma}{2},\,\frac{p'\gamma}{2}} \longleftrightarrow \mathfrak{B}_{1\to\gamma}\,. \tag{5.7}$$

One may be worried by the fact that the left-hand side of the dictionary seems to be dependent of the position on the quiver diagram, while the string vertex operator one the right-hand side doesn't carry any associated label. What happens is that the D-terms coming from the anomalous U(1)'s will give $p-1$ constraints relating the p different baryonic operators of (5.1), see [16]. Another issue is whether the gauge-theory chiral operator is a composite baryon $\mathfrak{B}_{1\to\gamma}$ or a product of elementary baryons $\mathfrak{B}_{\ell\to\ell+1}$. These two options differ from terms involving lower powers of the elementary baryons, using

JHEP04(2006)029

relations like (5.3a) between the composite and elementary baryons. We should also not forget that, as for the untwisted sector, there is some operator mixing at the level of the holographic dictionary. Anyway we get a prediction from the string theory side of the scaling dimensions of the baryonic chiral operators at the superconformal fixed point

$$\Delta(\mathfrak{B}_{1\to\gamma}) \;=\; \frac{4p'\gamma}{pp'+2}\,. \tag{5.8}$$

The operator with $\gamma = p$, i.e.

$$\mathfrak{B}_{1\to 1} = \det Q_{1\to 2}Q_{2\to 3}\cdots Q_{p\to 1}\,, \tag{5.9}$$

is given on the string theory side by the $\mathcal{N}=2$ Liouville potential. It can viewed either as belonging to the twisted or to the untwisted sector from the string theory point of view, which is consistent via the holographic dictionary to the chiral ring that we discussed above. This analysis carries over straightforwardly to the operators in the R-R sector. The R-R ground states of the untwisted sector correspond to the chiral operators (in $\mathcal{N}=1$ superfield notation) $\mathrm{Tr}\, W_1^\alpha (Q_{1\to 2}\cdots Q_{p\to 1})^r$.

The Argyres-Douglas point for which the curve is the most singular — or in other words for which the biggest number of dyons become massless — is obtained, taking for example an $SU(3)^p$ quiver, whose curve depends on the polynomial $P_3(x) = x^3 - \tilde{u}x - \tilde{v}$, by the hypersurface

$$\begin{aligned}
\tilde{u} &= \frac{1}{2}\mathrm{Tr}\,\langle \Xi^2\rangle = \frac{1}{2}\langle \mathfrak{L}_2\rangle - \frac{1}{6}\langle \mathfrak{L}_1^3\rangle = 0\,,\\
\tilde{v} &= \frac{1}{3}\mathrm{Tr}\,\langle \Xi^3\rangle = \langle \mathfrak{B}_{1\to 1}\rangle + \frac{1}{6}\langle \mathfrak{L}_1\mathfrak{L}_2\rangle - \frac{5}{54}\langle \mathfrak{L}_1^3\rangle = \pm 2\Lambda^{pp'}\,.
\end{aligned} \tag{5.10}$$

According to our dictionary, the double scaling limit of the string dual that we consider corresponds to the submanifold of the moduli space for which the only operator with non-zero vacuum expectation value away from the AD point is $\mathfrak{B}_{1\to 1}$, giving rise to the $\mathcal{N}=2$ Liouville perturbation in the string theory.

5.2 Dyon spectrum and stable D-branes

The various singularities of the SW curve of the $\mathcal{N}=1$ gauge theory correspond to dyons becoming massless as for the $\mathcal{N}=2$ theories. These dyons are of course not BPS anymore, and their domain of stability is unknown; nevertheless in the double scaling limit we describe a small neighborhood of the singularity where we can assume that they are stable. In the string theory dual they are naturally related to the localized D-branes in the $\mathcal{N}=1$ little string theory background.

These D-branes can be constructed in close parallel with the $\mathcal{N}=2$ models, because the orbifold acts freely. Then the one-point function on the disc for localized supersymmetric D-branes is the same as (3.12), up to an overall normalization, however now the closed string operators that can appear are restricted by the orbifold condition. In other words, the boundary state is constructed from a restricted set of Ishibashi states. The twisted sectors, containing the baryonic operators in the massless NS-NS spectrum, don't couple to the D-brane, so we have simply to keep the closed string operators with $m = 0 \mod p$. Then we can compute the open string partition function

$$Z_{\text{OPEN}}(\tau) = \frac{q^{\left(\frac{\hat{\mathbf{y}}'-\hat{\mathbf{y}}}{2\pi}\right)^2}}{\eta^2} \sum_{\{v_i\}\in(\mathbb{Z}_2)^3} \frac{1}{2} \sum_{a,b\in\mathbb{Z}_2} (-)^{a+b(1+v_1+v_2+v_3+m)} \frac{\Theta_{a+2v_1+\hat{s}_1-\hat{s}'_1,2}}{\eta} \times$$
$$\times \sum_{2j=0}^{n-2} \sum_{m\in\mathbb{Z}_{2n}} N^{j}_{\hat{\jmath}\hat{\jmath}'} \sum_{\omega\in\mathbb{Z}_p} C^{j\,(a+2v_3+\hat{s}_3-\hat{s}'_3)}_{2m+a+\hat{m}-\hat{m}'+2p'\omega}\; Ch_{\mathbb{I}}^{a+2v_2+\hat{s}_2-\hat{s}'_2}(m)\,. \tag{5.11}$$

We see that spacetime supersymmetry is broken by the sectors with $\omega \neq 0$. Also because of them the $\mathbb{Z}_{pp'}$-valued label $\hat{m}$ of the brane is now restricted to $\hat{m} \in \mathbb{Z}_{2p'}$ (otherwise it gives equivalent open string sectors).

Let's consider open strings with both endpoints on the same D-brane. The first type of strings are given by $v_1 = 1$ and $v_2 = v_3 = 0$. Then the sector $\omega = 0$ contains massless states in spacetime, corresponding to a four-dimensional gauge field reduced to 0+1 dimension, as in the $\mathcal{N} = 2$ case (3.15). In the Ramond sector we have similarly a four-dimensional massless spinor; altogether they form an $\mathcal{N} = 1$ gauge multiplet reduced to 0+1 dimensions. The sectors $\omega \neq 0$ for $v_1 = 1$ are all massive. However we have potentially new types of light states with a fermionic excitation along $SU(2)/U(1)$, i.e. $v_3 = 1$ and $v_1 = v_2 = v_4 = 0$. In all the sectors with $\omega \neq 0$, one can construct a worldsheet chiral primary of the worldsheet $\mathcal{N} = 2$ SCA, provided the open string spectrum contains a representation of spin j which satisfies $j + 1 = p'\omega$. Then for $m = 0$ we find a open string physical state with mass

$$m^2_{\text{TACH}} = -\frac{\omega}{\alpha' p}\,, \tag{5.12}$$

thus it is an open string tachyon and this D-brane will decay. To avoid these tachyons in the various open string sectors with both endpoints on the same D-brane, one needs to consider only the subset of D-branes with $SU(2)$ labels

$$\hat{\jmath} \leqslant \frac{p'}{2} - 1\,, \tag{5.13}$$

such that from the fusion rules $N^{j}_{\hat{\jmath}\hat{\jmath}'}$ we never have representations with $j+1 \geqslant p'$. One can check that for values of j lower than p' there are no tachyons either for the non-chiral states of $SU(2)/U(1)$. We conclude that the spectrum of open strings with both ends on any D-brane of this truncated set is non-tachyonic, thus they are stable in string theory. These results can be understood geometrically, see figure 3. At the level of the localized D-branes construction, the action of the orbifold on the $SU(2)/U(1)$ coset can be thought as a $2\pi/p$ rotation around the center of the coset target space metric, which is conformal to a disc; the fundamental domain of the orbifold contains p' of the "special points" on the boundary of the disc on which the D1-branes of the coset can end. The D-branes with $\hat{\jmath} \leqslant p'/2 - 1$ fit into the fundamental domain of the orbifold, so they won't intersect with their images and the open string sectors between different images — that break supersymmetry — will be very massive. On the contrary for D-branes that don't fit in the fundamental domain

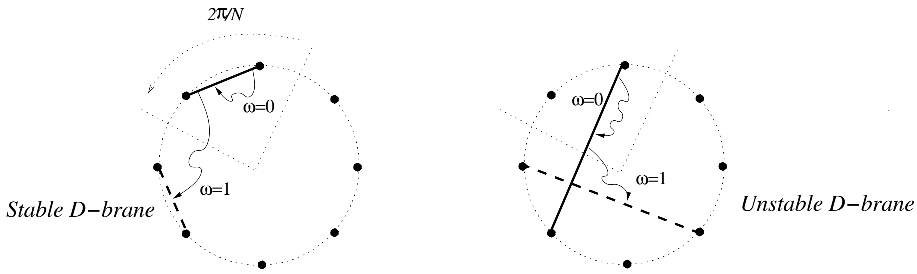

Figure 3: Stable D-brane (left) and unstable one (right) in $\mathcal{N}=1$ non-critical strings.

there is an unstable mode associated with the recombination of the intersections with their images.

This is compatible with the gauge theory expectations. Indeed since the Seiberg-Witten curve describing the Coulomb branch of the $\hat{A}_{p-1}$, $SU(p')$ quiver is quite similar to the curve of the $\mathcal{N}=2$ pure $SU(p')$ theory, thus the theory must contain the same number of light, stable dyons near the Argyres-Douglas point. The truncation of the allowed D-branes in the $\mathcal{N}=1$ model (5.13), together with the restriction $\hat{m} \in \mathbb{Z}_{2p'}$ already discussed, gives precisely the same number of D-branes as the $\mathcal{N}=2$ non-critical string dual to the $\mathcal{N}=2$ $SU(p')$ theory. We have seen also that the massless degrees of freedom on the D-branes correspond to an $\mathcal{N}=1$ gauge multiplet; by considering sectors of open strings between different stable D-branes we can compute the open string Witten index that will be the same as for D-branes in the dual of the $\mathcal{N}=2$ $SU(p')$ theory. This is not surprising since the charges of the dyons becoming massless can be computed from the monodromies of the Seiberg-Witten curve, which is the same for both gauge theories.

From the one-point function with a graviton vertex operator we can again compute the masses of these dyons. The value of the cigar perturbation is related to the masses of these stable localized D-branes which are the lightest non-perturbative states of the string theory, as in the $\mathcal{N}=2$ models. However, from the gauge theory side, these masses are not given anymore by the Seiberg-Witten curve of the theory since the dyons are not BPS. From the string theory dual we can see that the masses have the following scaling relation with respect to the moduli space coordinate $\tilde{v}$ of the $\mathcal{N}=1$ quiver corresponding to the $\mathcal{N}=2$ Liouville perturbation:

$$m^2_{\text{DYON}} \sim (\delta\tilde{v})^{\frac{pp'+2}{pp'}}\,, \tag{5.14}$$

which depends only on the product pp'. Although the physics of this $\mathcal{N}=1$ quiver in the Coulomb phase is quite similar to an $\mathcal{N}=2$ $SU(p')$ theory, we see that the scaling relation between the dyon mass and the deformation of the singularity is different. We can compute from the string theory dual the ratio of masses for dyons of labels $\hat{\jmath}$ and $\hat{\jmath}'$:

$$\frac{m_{\hat{\jmath}}}{m_{\hat{\jmath}'}} = \frac{\sin\frac{\pi(2\hat{\jmath}+1)}{pp'}}{\sin\frac{\pi(2\hat{\jmath}'+1)}{pp'}} \quad \text{for} \qquad 0 \leqslant \hat{\jmath},\, \hat{\jmath}' \leqslant \frac{p'}{2}-1\,. \tag{5.15}$$

This is a prediction of the string theory dual that cannot be matched to a similar quantity in the gauge theory, since the tools for computing it are lacking.[14] However, as again these ratios are independent of the double scaling parameter we expect that they are exact results in the gauge theory.

6. Conclusions

In this work we studied some non-critical string duals of four-dimensional supersymmetric gauge theories that are exactly solvable worldsheet conformal field theories. As for all the linear dilaton backgrounds the dual, non-gravitational theory is not a field theory, rather a little string theory, whose low-energy physics is nevertheless well described by a certain gauge theory. It is not possible to have a neat separation between the energy scales of the gauge theory and the energy scale of the little string theory — above which non-field theoretic excitations mix with the gauge theory — while keeping the string coupling constant small. Nevertheless worldsheet quantities having to do with the localized states (loosely speaking, excitations at the tip of the cigar) should correspond to field theory physics, and indeed are in good agreement with field theory expectations.

The first examples, that were already studied [2, 3, 28], of such models are $\mathcal{N}=2$ non-critical string duals of $SU(n)$ gauge theories near an Argyres-Douglas point where the field theory is superconformal. The string/gauge duality then allowed to compute the anomalous dimensions of chiral operators at the superconformal fixed point using the partition function of the string theory that we computed. We constructed explicitly the BPS, localized D-branes that are dual to the BPS dyons of the gauge theory at the superconformal fixed point, and show that their masses can be computed in terms of the coordinates of the moduli space of the Coulomb branch using the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1) - \mathcal{N}=2$ Liouville duality which is a worldsheet instanton effect. In other words this shows that worldsheet non-perturbative effects give the non-trivial Seiberg-Witten solution of the gauge theory.

We have shown that this duality between non-critical strings and gauge theories can be extended to theories with only $\mathcal{N}=1$ supersymmetry. More precisely, chiral orbifolds of the $\mathcal{N}=2$ non-critical strings considered above give $\mathcal{N}=1$ chiral quivers in the Coulomb phase, whose moduli space contains also Argyres-Douglas superconformal fixed points. These are to our knowledge the first examples of string duals of $\mathcal{N}=1$ chiral gauge theories that can be solved exactly, at the level of string perturbation theory. The spectrum of scaling dimensions for the chiral ring operators at the superconformal fixed point can be also computed in those examples. It is very interesting also that the dyons that become massless at the superconformal fixed point are in one-to-one correspondence with non-supersymmetric D-branes which are nevertheless stable. Indeed their massless spectrum is supersymmetric in spacetime. The predictions of the string duals for the masses of those dyons are quite interesting since similar quantities cannot be computed in the field theory.

There are obvious generalizations of these models, if we consider the more generic class of four-dimensional non-critical strings (2.1) with two $\mathcal{N}=2$ minimal models. These

[14] It is natural in the related context of AdS/CFT, where we consider orbifolds of $\mathcal{N}=4$ SYM, that the projected theory shares many properties with the original one.

JHEP04(2006)029

are dual to $\mathcal{N}=2$ quivers, and further orbifoldization leads to more complicated $\mathcal{N}=1$ quivers with interesting structure. We will study them in a forthcoming publication. Another interesting feature of these models is that, since they are intrinsically ten-dimensional superstring theories, they have a clear low-energy supergravity limit. From all these examples we see an interesting relation between critical phenomena in two dimensions, i.e. two-dimensional CFTs, and critical points of four-dimensional gauge theories. It would be interesting to understand how far this correspondence can be extended.

Another development of this work can be to construct, from both types of non-critical strings, four-dimensional $\mathcal{N}=1$ gauge theories by adding D-branes extended in the four spacetime dimensions, that are T-dual (along the flat space directions) to the "dyonic" D-branes discussed above, generalizing the construction with D-branes in the conifold [44, 45]. The D-branes of the $\mathcal{N}=1$ non-critical strings give in particular the possibility of obtaining supersymmetric gauge theories from non-supersymmetric D-branes.

The most interesting and challenging issue is to understand how to add tree level superpotentials to the gauge theories considered in this paper. From the string theory point of view it should correspond to adding non-normalizable marginal deformations to the worldsheet action, which raises many puzzles. If we trust the field theory picture at the level of the string theory — even though the non-critical string is not strictly speaking dual to the gauge theory — we expect to find a large variety of interesting phenomena. Indeed, from the gauge theory point of view, adding a tree level superpotential to an $\mathcal{N}=2$ or $\mathcal{N}=1$ theory at a critical point in the Coulomb phase can lead either to a lifting of this vacua, confinement through condensation of monopoles or an $\mathcal{N}=1$ superconformal field theory. These three possibilities would correspond respectively in the string theory to an instability (leading to a time-dependent solution pushing the theory away from the critical point), condensation of D-branes generating Ramond-Ramond flux and, if we start with the $\mathcal{N}=2$ non-critical strings, breaking of $\mathcal{N}=2$ to $\mathcal{N}=1$ by a marginal non-normalizable deformation.

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A. Representations and characters of worldsheet $\mathcal{N}=2$ superconformal algebra

We gather in this appendix some conventions and modular properties of characters that we use abundantly in the bulk of the paper, and in particular in the transformations of the annulus amplitude from open to closed string channels.

Free fermions

We define the theta-functions as:

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\tau,\nu)=\sum_{n\in\mathbb{Z}}q^{\frac{1}{2}(n+\frac{a}{2})^2}e^{2i\pi(n+\frac{a}{2})(\nu+\frac{b}{2})},$$

where $q=e^{2\pi i\tau}$. It will be convenient to split the states inside the R and NS sectors according to their fermion number:

$$\begin{aligned}\frac{1}{2\eta}\left\{\vartheta\begin{bmatrix}0\\0\end{bmatrix}-\vartheta\begin{bmatrix}0\\1\end{bmatrix}\right\}&=\frac{\Theta_{0,2}}{\eta}\\ \frac{1}{2\eta}\left\{\vartheta\begin{bmatrix}0\\0\end{bmatrix}+\vartheta\begin{bmatrix}0\\1\end{bmatrix}\right\}&=\frac{\Theta_{2,2}}{\eta}\\ \frac{1}{2\eta}\left\{\vartheta\begin{bmatrix}1\\0\end{bmatrix}-i\vartheta\begin{bmatrix}1\\1\end{bmatrix}\right\}&=\frac{\Theta_{1,2}}{\eta}\\ \frac{1}{2\eta}\left\{\vartheta\begin{bmatrix}1\\0\end{bmatrix}+i\vartheta\begin{bmatrix}1\\1\end{bmatrix}\right\}&=\frac{\Theta_{3,2}}{\eta}\end{aligned}\tag{A.1}$$

in terms of the theta functions of $\hat{\mathfrak{su}}(2)$ at level 2:

$$\Theta_{m,k}(\tau,\nu)=\sum_{n\in\mathbb{Z}}q^{k\left(n+\frac{m}{2k}\right)^2}e^{2i\pi\nu k\left(n+\frac{m}{2k}\right)}.$$

The modular transformation property of the fermionic characters is then:

$$\frac{\Theta_{s,2}(-1/\tau,\nu/\tau)}{\eta(-1/\tau)}=\frac{1}{2}e^{i\pi\nu^2/\tau}\sum_{s'\in\mathbb{Z}_4}e^{-i\pi ss'/2}\frac{\Theta_{s',2}(\tau,\nu)}{\eta(\tau)}.\tag{A.2}$$

We will often work in terms of these characters for fermions in NS or R sector, projected onto even or odd fermion number states.

$\mathcal{N}=2$ minimal models

The $\mathcal{N}=2$ minimal models correspond to the supersymmetric gauged WZW model $SU(2)_k/U(1)$, and are characterized by the level k of the supersymmetric WZW model. The $\mathcal{N}=2$ minimal models characters are determined implicitly through the identity:

$$\sum_{m\in\mathbb{Z}_{2k}}\mathcal{C}^{j\,(s)}_{m}\Theta_{m,k}=\chi^j\Theta_{s,2}\,,\tag{A.3}$$

where χ^j denotes a character of bosonic $SU(2)$ at level $k-2$. The characters are labeled by the triplet (j,m,s). They correspond to the primaries of the coset $[SU(2)_{k-2}\times SO(2)_1]/U(1)_k$, however are not generically primaries of the $\mathcal{N}=2$ algebra. They have the R-charge

$$Q=\frac{s}{2}-\frac{m}{k}\mod 2\tag{A.4}$$

The following identifications apply:

$$
\begin{aligned}
(j,m,s) &\sim (j,m+2k,s)\\
(j,m,s) &\sim (j,m,s+4)\\
(j,m,s) &\sim (k/2-j-1,m+k,s+2)
\end{aligned}
$$

as well as the selection rule

$$2j+m+s=0 \mod 2 \tag{A.5}$$

The weights of the primaries states are as follows:

$$
\begin{array}{lll}
h = \frac{j(j+1)}{k} - \frac{n^2}{4k} + \frac{s^2}{8} & \text{for} & -2j \leqslant n-s \leqslant 2j \\
h = \frac{j(j+1)}{k} - \frac{n^2}{4k} + \frac{s^2}{8} + \frac{n-s-2j}{2} & \text{for} & 2j \leqslant n-s \leqslant 2k-2j-4
\end{array}
\tag{A.6}
$$

We have the following modular S-matrix for these characters:[15]

$$S^{j\,m\,s}{}_{j'\,m'\,s'} = \frac{1}{2k}\sin\pi\frac{(1+2j)(1+2j')}{k}\; e^{i\pi\frac{mm'}{k}}\; e^{-i\pi ss'/2}. \tag{A.7}$$

Note also that the fusion rules of $SU(2)$ are given by:

$$N_{\hat{\jmath}\hat{\jmath}'}^{j} = 1 \text{ for } |\hat{\jmath}-\hat{\jmath}'| \leqslant j \leqslant \min\{\hat{\jmath}+\hat{\jmath}', k-\hat{\jmath}-\hat{\jmath}'\} \text{ and } j+\hat{\jmath}+\hat{\jmath}' \in \mathbb{Z}\ , \quad 0 \text{ otherwise.} \tag{A.8}$$

Supersymmetric $SL(2,\mathbb{R})/U(1)$

The characters of the $SL(2,\mathbb{R})/U(1)$ super-coset at level k come in different categories corresponding to the classes of irreducible representations of the $SL(2,\mathbb{R})$ algebra in the parent theory. These coset characters coincide with the characters of the irreducible representations of the $\mathcal{N}=2$ superconformal algebra with $c>3$. In all cases the quadratic Casimir of the representations is $c_2 = -j(j-1)$. First we consider *continuous representations*, with $j = 1/2 + ip$, $p \in \mathbb{R}^+$. The characters are denoted by $ch_c(p,m)\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$, where the $N=2$ superconformal $U(1)_R$ charge of the primary is $Q = 2m/k$. Then we have *discrete representations* with $1/2 < j < (k+1)/2$, of characters $ch_d(j,r)\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$, where the $\mathcal{N}=2$ $U(1)_R$ charge of the primary is $Q = 2(j+r+a/2)/k$, $r \in \mathbb{Z}$. The third category corresponds to the *finite representations*, with $j = (u-1)/2$ and where $u = 1,2,3,\ldots$ denotes the dimension of the finite representation. These representations are not unitary except for the trivial representation with $u=1$. The character for this identity representation we denote by $ch_{\mathbb{I}}(r)\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$. We can also define characters labeled by a $\mathbb{Z}_4$ valued quantum number for $SL(2,\mathbb{R})/U(1)$, following the method we used to define these characters for the free fermions. In other words, we define these characters by summing over untwisted and twisted NS or R sectors with the appropriate signs.

[15]The reader may notice that the S-matrix of the N=2 minimal model given here may differ by a factor of two with the literature. Indeed in our conventions, the S-matrix is defined as acting on all triplets (j,n,s), while it is often defined as acting on the fundamental domain w.r.t. the identification $(j,n,s) \sim (k/2-j-1,n+k,s+2)$.

JHEP04(2006)029

It is often convenient to define *extended characters*, provided that k is rational. In our particular example $k = 2n/(n+2)$, and they are defined by summing over by summing over $2n$ units of spectral flow.[16] These characters correspond to an extended chiral algebra which can be constructed along the line of the extended chiral algebra for a $U(1)$ boson at rational radius squared. For example, for the continuous characters we define the corresponding extended characters (denoted by capital letters) by:

$$Ch_c(P,M)\begin{bmatrix}a\\b\end{bmatrix} = \sum_{w\in\mathbb{Z}} ch_c\left(P,\frac{M}{2(n+2)}+2nw\right)\begin{bmatrix}a\\b\end{bmatrix} = \frac{q^{\frac{(n+2)P^2}{2n}}}{\eta^3}\,\Theta_{M,2n(n+2)}\,\vartheta\begin{bmatrix}a\\b\end{bmatrix}. \qquad \text{(A.9)}$$

They carry a $\mathbb{Z}_{4n(n+2)}$ charge given by $M \in \mathbb{Z}$. In our conventions the J_3 eigenvalue of $\mathrm{SL}(2,\mathbb{R})$ is $m = {}^{M}/_{2(n+2)}$. In contrast with standard characters, their modular transformations involve only a discrete set of $N = 2$ charges.

Let us now consider briefly the discrete representations. They appear in the closed string spectrum in the range $1/2 < j < (k+1)/2$. The primary states are:

$$\begin{aligned}
|j,m=j+r\rangle &= |0\rangle_{\mathrm{NS}} \otimes |j,m=j+r\rangle_{\mathrm{SL}(2,\mathrm{R})} \quad r \geqslant 0\\
|j,m=j+r\rangle &= \psi^-_{-\frac{1}{2}}|0\rangle_{\mathrm{NS}} \otimes (J^-_{-1})^{-r-1}|j,j\rangle_{\mathrm{SL}(2,\mathrm{R})} \quad r < 0
\end{aligned}$$

Thus in the $\mathbb{Z}_4$ formalism the former are in the $s = 0$ sector and the latter in the $s = 2$ sector. These NS primary states have weights

$$\begin{aligned}
h &= \frac{j(2r+1)+r^2}{k} & r \geqslant 0\\
h &= \frac{j(2r+1)+r^2}{k} - r - \frac{1}{2} & r < 0
\end{aligned}$$

To define conveniently the extended discrete characters, we adopt the following notation:

$$Ch_d\begin{bmatrix}a\\b\end{bmatrix}(j,M) = \sum_{w\in\mathbb{Z}} ch_d\left(P,\frac{M}{2(n+2)}+2nw-j-\frac{a}{2}\right)\begin{bmatrix}a\\b\end{bmatrix} \qquad \text{(A.10)}$$

with the convention that this character is zero whenever $\frac{M}{2(n+2)} + 2nw - j - \frac{a}{2}$ is not an integer. We don't need the explicit expression of these characters for our purposes; it can be found e.g. in [39]

To compute the annulus amplitude between localized D-branes in the bulk of the text, we need the modular transformation of the extended character associated to the trivial representation (in the $\mathbb{Z}_4$ formalism for fermions) that appears in the open string spectrum.

[16] If n is even then one can reduce further the characters by summing over n units of spectral flow.

JHEP04(2006)029

It is given by [38, 39]

$$
\begin{aligned}
Ch_{\mathbb{I}}^{(s)}(r;-1/\tau,0) &= \sum_w ch_{\mathbb{I}}^{(s)}(r+2nw;-1/\tau,0) \qquad \text{(A.11)}\\
=& \frac{1}{2n}\sum_{s'\in\mathbb{Z}_4} e^{-i\pi\frac{ss'}{2}}\left\{\int_0^\infty \mathrm{d}p' \sum_{M'\in\mathbb{Z}_{4n(n+2)}} e^{-i\pi\frac{rM'}{n}}\frac{\sinh 2\pi p' \sinh[\pi(n+2)p'/n]}{\cosh 2\pi p' + \cos\pi\left(\frac{M'}{n+2}-s'\right)} Ch_c^{(s')}(p',M';\tau,0)\right.\\
&\left. + \sum_{2(n+2)j'=n+3}^{3n+1}\sum_{r'\in\mathbb{Z}_n} y\sin\left(\frac{(n+2)\pi}{2n}(2j'-1)\right) e^{-2i\pi\frac{(n+2)(j'+r')r}{n}} Ch_d^{(s')}(j',r',\tau,0)\right\}
\end{aligned}
$$

The spectrum of primaries in the NS sector for this identity representation is as follows. First we have the identity operator $|r=0\rangle_{\mathrm{SL(2,R)}}\otimes|0\rangle_{\mathrm{NS}}$ belonging to the sector $s=0$. The other primary states are:

$$
\begin{aligned}
|r\rangle &= \psi^+_{-\frac{1}{2}}|0\rangle_{\mathrm{NS}}\otimes(J^+_{-1})^{r-1}|r=0\rangle_{\mathrm{SL(2,R)}} \quad r>0\\
|r\rangle &= \psi^-_{-\frac{1}{2}}|0\rangle_{\mathrm{NS}}\otimes(J^-_{-1})^{-r-1}|r=0\rangle_{\mathrm{SL(2,R)}} \quad r<0
\end{aligned}
$$

They belong to the sector $s=2$ and have weights

$$
\begin{aligned}
h &= \frac{r^2}{k}+r-\frac{1}{2} \quad r>0\\
h &= \frac{r^2}{k}-r-\frac{1}{2} \quad r<0
\end{aligned}
$$

the Ramond sector is obtained by one-half spectral flow. In particular, r will be half-integer.

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